On the eigenvalues of a polyharmonic matrix operator near diffraction planes

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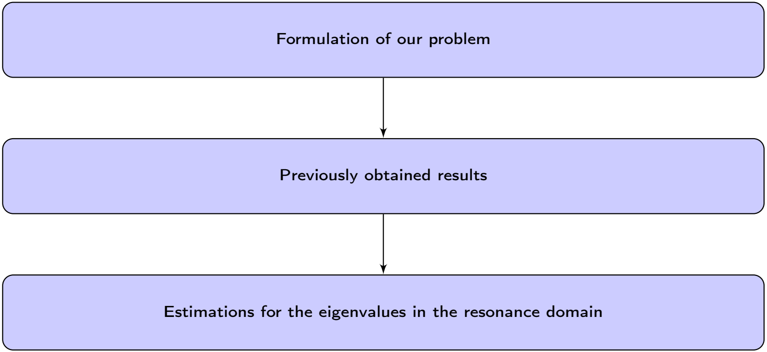
Motivation

The organization of this paper is as follows:



For $\frac{1}{2} < l < 1$, we consider the operator

$$H(l,V)u(x) = (-\boldsymbol{\Delta})^l u(x) + V(x)u(x) \tag{1}$$

in $L_2^s(\boldsymbol{R}^d)$, where $(-\boldsymbol{\Delta})^l$ is a diagonal $s \times s$ matrix whose diagonal elements are scalar polyharmonic operators $(-\Delta)^l$,
$V(x) = (v_{ij}(x))$, $i,j = 1,2,\ldots,s$, is a real-valued symmetric $s \times s$ matrix, $s \geq 2$, $x = (x_1, x_2, \ldots, x_d) \in \boldsymbol{R}^d$, $d \geq 2$.

Let Ω be an arbitrary lattice and $K \equiv \mathcal{R}^d/\Omega$ be a fundamental domain of Ω then the dual lattice of Ω is

$$\Gamma = \{\gamma \in \boldsymbol{R}^d : (\gamma, w) \in 2\pi\boldsymbol{Z}, \forall w \in \Omega\}$$

and $K^* \equiv \boldsymbol{R}^d/\Gamma$ is its fundamental domain.

It is well known that the spectral analysis of $H(l, V)$ can be reduced to studying the eigenvalues of the operators $H_t(l, V)$, for all $t \in K^*$, given by the differential expression

$$H_t(l, V)u(x) = (-\boldsymbol{\Delta})^l u(x) + V(x)u(x) \qquad (2)$$

in $L_2^s(K)$ and the quasiperiodic condition

$$u(x + w) = e^{iw \cdot t} u(x), w \in \Omega, \quad t \in K^*$$

$$u(x) = (u_1(x), u_2(x), \cdots, u_s(x)), x \in K.$$

($\cdot$ denotes the inner product in $\boldsymbol{R}^d$)

That is; the spectrum of the operator $H_t(l,V)$ consists of the eigenvalues

$$\Lambda_1(t) \leq \Lambda_2(t) \leq \dots \quad \text{and}$$

$$spec(H(l,V)) = \bigcup_{N=1}^{\infty} \{\Lambda_N(t) : t \in K^*\},$$

where $\Lambda_N(t)$ are the eigenvalues of $H_t(l,V)$ and we denote the corresponding eigenfunctions of $H_t(l,V)$ by $\Psi_{N,t}(x)$.

When $V(x) = 0$ in (2) denote the operator defined by (2) by $H_t(l, 0)$. The eigenvalues of the unperturbed operator $H_t(l, 0)$ are $\mid \gamma + t \mid^{2l}$ and the corresponding eigenspaces are

$$E_{\gamma,t} = span\{\Phi_{\gamma,t,1}(x), \Phi_{\gamma,t,2}(x), \ldots, \Phi_{\gamma,t,s}(x)\},$$

$$\Phi_{\gamma,t,j}(x) = (0, \ldots, 0, e^{i(\gamma+t)\cdot x}, 0, \ldots, 0),$$

$j = 1, 2, \ldots, s$ for $\gamma \in \Gamma$, $t \in K^*$.

We note that the non-zero component $e^{i(\gamma+t)\cdot x}$ of $\Phi_{\gamma,t,j}(x)$ stands in the j-th component.

The Goal

To obtain the asymptotic formulas for the eigenvalues of $H(l, V)$ for arbitrary $\frac{1}{2} < l < 1$ which correspond to **the resonance eigenvalues** of the unperturbed operator $H(l, 0)$ (when $V(x) = 0$ in (1)), roughly speaking when they lie near planes of diffraction

$$\left\{ x \in R^d : |x|^{2l} = |x + b|^{2l} \right\}.$$

We use the same method introduced by Veliev in his papers [18,20,24] and define the following parameters:

$$\alpha(l) = \frac{a}{(d+20)3^{d+1}} \ ,$$

$$\alpha_k(l) = 3^k \alpha(l), \ \ k = 1, 2, \dots, d-1,$$

where $l = \frac{1}{2} + a$, $0 < a < \frac{1}{2}$.

Resonance Eigenvalues of $H_t(L,0)$

Let $m > \frac{(4d-1)}{2}(d+20)3^{d+1} + \frac{d}{4}3^d + d + 1$, $p = m - d$, ρ is a large parameter,
$\Gamma(\rho^{\alpha(l)}) \equiv \left\{\gamma \in \frac{\Gamma}{2} : 0 <| \gamma + t |< \rho^{\alpha(l)}\right\}$.

As in the papers of Veliev, we divide the eigenvalues $|\gamma + t|^{2l}$ of the unperturbed operator into two domains:

Resonance Domain

$$V_b^l(\rho^{\alpha_1(l)}) \equiv \left\{x \in R^d : \left||x|^{2l} - |x+b|^{2l}\right| < \rho^{\alpha_1(l)}\right\}$$

for $b \in \Gamma(p\rho^{\alpha(l)})$

- The eigenvalue $|\gamma + t|^{2l}$ is called a resonance eigenvalue if $\gamma + t \in V_b^l(\rho^{\alpha_1(l)})$.

Non- Resonance Domain

$$U^l(\rho^{\alpha_1(l)}, p) \equiv \mathbb{R}^d \setminus E_1^l(\rho^{\alpha_1(l)}, p),$$

$$E_1^l(\rho^{\alpha_1(l)}, p) \equiv \bigcup_{b \in \Gamma(p\rho^{\alpha(l)})} V_b^l(\rho^{\alpha_1(l)})$$

- The eigenvalue $|\gamma + t|^{2l}$ is called a non-resonance eigenvalue if $\gamma + t \in U^l(\rho^{\alpha_1(l)}, p)$.

Also we define the set $E_k^l(\rho^{\alpha_k(l)}, p) = \bigcup_{\gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma(p\rho^{\alpha(l)})} \left(\bigcap_{i=1}^{k} V_{\gamma_i}^l(\rho^{\alpha_k(l)})\right)$.

Remark

The non-resonance domain $U^l(\rho^{\alpha_1(l)}, p)$ has asymptotically full measure in $\boldsymbol{R}^d$ in the sense that $\frac{\mu(U^l(\rho^{\alpha_1(l)}, p) \bigcap B(\rho))}{\mu(B(\rho))} \to 1$, as $\rho \to \infty$, where $B(\rho) = \{x \in \boldsymbol{R^d} : \mid x \mid = \rho\}$, if

$$\alpha_1(l) - 2l + 2 + d\alpha(l) < 1 - \alpha(l) \quad \text{for} \quad \frac{1}{2} < l < 1$$

holds and the domain $V_b^l(\rho^{\alpha_1(l)}) \backslash E_2^l$, called a single resonance domain, has asymptotically full measure on $V_b^l(\rho^{\alpha_1(l)})$, that is, $\frac{\mu((V_b^l(\rho^{\alpha_1(l)}) \backslash E_2^l) \bigcap B(\rho))}{\mu(V_b^l(\rho^{\alpha_1(l)}) \bigcap B(\rho))} \to 1$, as $\rho \to \infty$, if

$$2\alpha_2(l) - 2l + 2 - \alpha_1(l) + (d+3)\alpha(l) < 1 \quad \text{for} \quad \frac{1}{2} < l < 1$$

holds.

Previously Obtained Results

Due to its physical importance, the most significant progress has been achieved in the case of Schrödinger operator; i.e. the case $l = 1$ in (1).

- For the first time asymptotics formulas for the eigenvalues of the periodic Schrödinger operator are obtained in the papers [17,18,24] by Veliev.
- When this operator is considered with Dirichlet boundary conditions on 2-dimensional rectangle, the high energy asymptotics of the eigenvalues are obtained in [8].
- In papers [10,11], the formulas for the eigenvalues of the Schrödinger operator considered with Dirichlet and Neumann boundary conditions on a d-dimensional parallelepiped, for arbitrary $d \geq 2$ are obtained.

- The high energy asymptotics of the eigenvalues of $H(l, V)$ for $4l > d+1$ $(d \geq 2)$ are obtained by Karpeshina in [13], and for arbitrary $l \geq 1$ $(d \geq 2)$ by Veliev in [20], where he claimed that the assumption $l \geq 1$ can be replaced by $l > n_{m,d}$ for some number $n_{m,d} < 1$ that depends on m (the smoothness of $V(x)$) and d (the dimension) without giving any technical details.
- For the matrix case, $s \geq 1$, $d \geq 2$, $l \geq 1$, and $4l > d+1$, asymptotic formulas for the eigenvalues of the operator $H(l, V)$ are obtained in [14].
- Again for the matrix case, in the study [25], Karakılıç has obtained the asymptotic formulas for the non-resonance eigenvalues (roughly, the ones far away from the diffraction planes) of the operator $H(l, V)$ when $\frac{1}{2} < l < 1$, $(n_{m,d} = \frac{1}{2})$, $s \geq 2$, and $d \geq 2$.

Assumptions

In this study we assume that

- $\gamma \notin V_{e_k}^{l}(\rho^{\alpha_1(l)})$ for $k = 1, 2, \ldots, d,\ d \geq 2$, where
  - $e_1 = (\frac{\pi}{a_1}, 0, \ldots, 0), e_2 = (0, \frac{\pi}{a_2}, \ldots, 0), \ldots, e_d = (0, \ldots, 0, \frac{\pi}{a_d})$

  and
- $\mid \gamma + t \mid^{2l}$ is a **resonance eigenvalue**, of order ρ^{2l} (written as $\mid \gamma + t \mid \sim \rho$), of the operator $H_t(l, 0)$, that is, $\gamma \in (\bigcap_{i=1}^{k} V_{\gamma_i}^{l}(\rho^{\alpha_k(l)})) \setminus E_{k+1}^{l}$, $k = 1, 2, \ldots, d-1$, $\gamma_i \neq e_j$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, d-1$.

- We assume that each entry $v_{ij}(x)$ is periodic with respect to a lattice Ω and is a real valued function of $W_2^m(K)$,where $K \equiv \mathcal{R}^d/\Omega$ is a fundamental domain of Ω, and

$$m > \frac{(4d-1)}{2}(d+20)3^{d+1} + \frac{d}{4}3^d + d + 1.$$

Since $\{e^{i\gamma\cdot x}\}_{\gamma\in\Gamma}$ is complete in $L_2(K)$, each matrix element $v_{ij}(x) \in L_2(K)$ of the matrix $V(x)$ can be written as

$$v_{ij}(x) = \sum_{\gamma\in\Gamma} v_{ij\gamma} e^{i\gamma\cdot x}, \; v_{ij\gamma} = (v_{ij}(x), e^{i\gamma\cdot x}) = \int_K v_{ij}(x) e^{-i\gamma\cdot x} dx.$$

For each $i, j = 1, 2, \ldots, s$, $v_{ij}(x) \in W_2^m(K)$ means that

$$\sum_{\gamma\in\Gamma} \mid v_{ij\gamma} \mid^2 (1 + \mid \gamma + t \mid^{2m}) < \infty.$$

Moreover, for a big parameter, we can write

$$v_{ij}(x) = \sum_{\gamma \in \Gamma(\rho^{\alpha(l)})} v_{ij\gamma} e^{i\gamma \cdot x} + O(\rho^{-p\alpha(l)})$$

and define

$$M_{ij} \equiv \sum_{\gamma \in \Gamma} | v_{ij\gamma} | < \infty,$$

for all $i, j = 1, 2, \ldots, s$, where $p = m - d$, $\alpha > 0$ and

$$\Gamma(\rho^{\alpha(l)}) = \{\gamma \in \Gamma : 0 < | \gamma + t | < \rho^{\alpha(l)}\}.$$

If $\gamma = 0$, $v_{ij0} = \int_K v_{ij}(x)dx$ and

$$V_0 = (v_{ij0}) = \int_K V(x)dx$$

is a symmetric $s \times s$ matrix.

We define the following sets:

- $B_k(l) = \{b : b = \sum_{i=1}^{k} n_i\gamma_i, n_i \in \mathbb{Z}, \mid b \mid < \frac{1}{2}\rho(l)\}$,
- $B_k(l,\gamma) = \gamma + B_k(l) = \{\gamma + b : b \in B_k(l)\}$,
- $B_k(l,\gamma,p_1) = B_k(l,\gamma) + \Gamma(p_1\rho^{\alpha}(l))$,

where p_1 is the integer part of $\frac{p-1}{2}$ and

$$\rho(l) = \rho^{\frac{1}{2}\alpha_{k+1}(l)+(k-1)\alpha(l)-2l+2}.$$

- $b_k(l) :=$ the number of the vectors in $B_k(l,\gamma,p_1)$,
- $h_\tau :=$ the vectors of $B_k(l,\gamma,p_1)$ $(\tau = 1, 2, \ldots, b_k(l))$.

We define the $sb_k(l) \times sb_k(l)$ matrix $C = C_k(l, \gamma)$ as

$$C = \begin{bmatrix} \mid h_1 + t \mid^{2l} I - V_0 & V_{h_1 - h_2} & \cdots & V_{h_1 - h_{b_k(l)}} \\ V_{h_2 - h_1} & \mid h_2 + t \mid^{2l} I - V_0 & \cdots & V_{h_2 - h_{b_k(l)}} \\ \vdots & & & \\ V_{h_{b_k(l)} - h_1} & V_{h_{b_k(l)} - h_2} & \cdots & \mid h_{b_k(l)} + t \mid^{2l} I - V_0 \end{bmatrix},$$

where $V_{h_\tau - h_\xi}$ $(\tau, \xi = 1, 2, \ldots, b_k(l))$ is an $s \times s$ matrix

$$V_{h_\tau - h_\xi} = \begin{bmatrix} v_{11h_\tau - h_\xi} & v_{12h_\tau - h_\xi} & \cdots & v_{1sh_\tau - h_\xi} \\ v_{21h_\tau - h_\xi} & v_{22h_\tau - h_\xi} & \cdots & v_{2sh_\tau - h_\xi} \\ \vdots & & & \\ v_{s1h_\tau - h_\xi} & v_{s2h_\tau - h_\xi} & \cdots & v_{ssh_\tau - h_\xi} \end{bmatrix}.$$

Theorem 1.

Let $\gamma + t \in (\bigcap_{i=1}^{k} V_{\gamma_i}^{l}(\rho^{\alpha_k(l)})) \setminus E_{k+1}^{l}$ be a resonance eigenvalue of the operator $H_t(l,0)$, $k = 1,2,\ldots,d-1$, $\gamma_i \neq e_j$, $i = 1,2,\ldots,k$, $j = 1,2,\ldots,d-1$, and $\frac{1}{2} < l < 1$.

Then for each eigenvalue $\eta_m(l,\gamma)$ of the matrix C satisfying

$$\mid \eta_m(l,\gamma) - \mid \gamma + t \mid^{2l} \mid < \frac{3}{8}\rho^{\alpha_1(l)},$$

there is an eigenvalue $\Lambda_N(t)$ of the operator $H_t(l,V)$ such that

$$\Lambda_N(t) = \lambda_m(l,\gamma) + O(\rho^{-(p-\frac{d}{4}3^d)\alpha(l)+d-\frac{1}{2}}).$$

Theorem 2.

Let $\mid \gamma + t \mid^{2l}$ be a resonance eigenvalue of the unperturbed operator $H_t(l,0)$, that is, $\gamma + t \in (\bigcap_{i=1}^{k} V_{\gamma_i}^{l}(\rho^{\alpha_k(l)})) \setminus E_{k+1}^{l}$, $k = 1, 2, \ldots, d-1,\ \gamma_i \neq e_j,\ i = 1, 2, \ldots, k,\ \ j = 1, 2, \ldots, d-1,$ where $\mid \gamma + t \mid \sim \rho$, and $\Lambda_N(t)$ be an eigenvalue of the operator $H_t(l,V)$ for which the following condition

$$\mid \Lambda_N(t) - \mid \gamma + t \mid^{2l} \mid < \frac{1}{2}\rho^{\alpha_1(l)}$$

holds and its corresponding eigenfunction $\Psi_{N,t}$ satisfies

$$\mid < \Psi_{N,t}, \Phi_{\gamma,t,j} > \mid > c_4 \rho^{-c\alpha(l)}, \quad j = 1, 2, \ldots, s.$$

Then there exists an eigenvalue $\eta_m(l,\gamma)$, $1 \leq m \leq sb_k(l)$ of the matrix C such that

$$\Lambda_N(t) = \eta_m(l,\gamma) + O(\rho^{-(p-c-\frac{d}{4}3^d)\alpha(l)+\frac{d}{2}}).$$

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