Ground states for Beris – Edward model of liquid crystals.

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We study the Beris-Edward system of equations which describes the evolution in time of nematic liquid crystals
A.-N. Beris and B.-J. Edwards. Thermodynamics of flowing systems with internal microstructure. Oxford Engineering Science Series 36. Oxford, New York: Oxford university Press, 1994
and
M. Paicu and A. Zarnescu. "Global Existence and Regularity for the Full Coupled Navier-Stokes and Q-Tensor System". In: SIAM Journal on Mathematical Analysis 43.5(2011), pp. 2009 – 2049.

Nematic liquid crystal molecules are characterized by two important features: they can flow and they align along preferred directions. The mathematical tool used in this framework to capture the local orientation of the molecules is the so called Q -tensor
Apala Majumdar. "Equilibrium order parameters of nematic liquid crystals in the Landau-de Gennes theory". In: European Journal of Applied Mathematics 21.2(2010), pp. 181 – 203 .

The Q-tensor

Formally a Q -tensor is a function $Q(t,x)$ with values into the set of traceless symmetric matrices:

$$\mathcal{S}_0^{(n)} := \left\{ Q \in \mathbb{R}^{n\times n}_{Sym} \mid \mathrm{tr}(Q) = 0 \right\}.$$

The Beris-Edward system

The Beris-Edward system models the evolution of liquid crystal molecules together with the underlying flow. It couples an energy equation for the evolution of the Q -tensor with a forced Navier-Stokes equation for incompressible fluids describing the evolution of the flow velocity $u(t,x) \in \mathbb{R}^n$:

$$\begin{cases} (\partial_t + u \cdot \nabla) Q = \Gamma H + S(\nabla u, Q) \\ (\partial_t + u \cdot \nabla) u = \nu \Delta u + \nabla p + \nabla \cdot (\tau + \sigma) \\ \nabla \cdot u = 0 \end{cases}$$

where $(t,x) \in (0,T) \times \mathbb{R}^n, \Gamma, \nu > 0.$

The equation for the Q-tensor

$$H = L\Delta Q - aQ + b\left(Q^2 - \frac{\mathrm{tr}\left(Q^2\right)}{n} Id\right) - c\,\mathrm{tr}\left(Q^2\right) Q,$$

$$\Omega = \frac{\nabla u - (\nabla u)^t}{2}, \quad D = \frac{\nabla u + (\nabla u)^t}{2},$$

$$S(\nabla u, Q) = (\xi D + \Omega)\left(Q + \frac{1}{n} Id\right) + \left(Q + \frac{1}{n} Id\right)(\xi D - \Omega) - 2\xi\left(Q + \frac{1}{n} Id\right)\mathrm{tr}(Q\nabla u),$$

where $\xi \geq 0$.

The equation for the flux

$$\tau = -\xi \left(Q + \frac{Id}{n} \right) H - \xi H \left(Q + \frac{Id}{n} \right) + 2\xi \left(Q + \frac{Id}{n} \right) QH$$
$$- L \nabla Q \odot \nabla Q,$$
$$\sigma = QH - HQ.$$

Notations

Here and in what follows we will always assume the Einstein summation convention.

The notations adopted are the following:

- for a matrix $Q \in \mathbb{R}^{n \times n}$ we define its Frobenius norm $|Q| = \sqrt{QQ^t} = \sqrt{Q_{ij}Q_{ij}}$,
- for a function $Q(x)$ taking values from $\mathbb{R}^n$ into $\mathbb{R}^{n \times n}$ we formally set $\nabla Q(x)$ as the tensor of entries $(\nabla Q)_{ij,k} := \partial_k Q_{ij}$ for $0 \leq i,j \leq n, 1 \leq k \leq n$.

Notations

- $|\nabla Q| = \sqrt{\partial_k Q_{ij} \partial_k Q_{ij}}$, so that $|\nabla Q|^2 = \sum_{ij} |\nabla Q_{ij}|^2$ where ∇Q_{ij} is the standard gradient of the $(i, j)-$ the entry of Q,
- we formally denote as ΔQ the matrix in $\mathbb{R}^{n \times n}$ with entries $(\Delta Q)_{ij} = \Delta Q_{ij}$,
- we denote as $\nabla Q \odot \nabla Q$ the matrix in $\mathbb{R}^{n \times n}$ such that $(\nabla Q \odot \nabla Q)_{ij} = \partial_i Q_{hl} \partial_j Q_{hl}$.

The Beris-Edward system

We are interested in the case $\xi = 0$, furthermore we will be assuming $n = 3$ and we set $\Gamma = L = \nu = 1$.
With these assumptions the Beris-Edward system can be rewritten as:

$$\begin{cases} (\partial_t + u \cdot \nabla)\, Q = \Omega Q - Q\Omega + H \\ (\partial_t + u \cdot \nabla)\, u = \Delta u + \nabla p - \nabla \cdot (\nabla Q \odot \nabla Q) + \nabla \cdot (Q\Delta Q - \Delta Q Q) \\ \nabla \cdot u = 0 \end{cases}$$

Landau-de Gennes energy

A brief analysis shows that the equation for the Q -tensor, in the case of null flux $u = 0$, is nothing but a gradient flow for the Landau-de Gennes energy:

$$E(Q) := \int_{\mathbb{R}^n} \frac{1}{2}|\nabla Q|^2 + \frac{a}{2}\operatorname{tr}\left(Q^2\right) - \frac{b}{3}\operatorname{tr}\left(Q^3\right) + \frac{c}{4}\left[\operatorname{tr}\left(Q^2\right)\right]^2 dx$$

In fact the term H in the first equation is the first variation of the the Landau-de Gennes energy when considered as a function defined on the subspace $\mathcal{S}_0^{(n)}$.

Landau-de Gennes energy

The coefficients a, b, c found in the literature are usually taken to be as follows: $a, b \in \mathbb{R}, c > 0$.
In this work we will further assume that $a, b, c > 0$ are such that $\exists \zeta > 0$ such that $\frac{a}{2}\zeta^2 - \frac{b}{3}\zeta^3 + \frac{3c}{2}\zeta^4 < 0$.
We observe that such a condition is verified, for example, if the coefficients satisfy $b^2 > 27ac$.

Ground states

We address the problem of the existence of ground states for the Beris-Edward system and their stability for small perturbations. Driven by a work of Cavaterra, Rocca et al. *SIAM Journal on Mathematical Analysis*, 2016, we look for solutions of the form $(Q, 0)$ i.e. with no flux.
In that paper the authors, that worked in the periodic setting with $n = 2$, proved that the ω-limit set of a global solution $(Q(x, t), u(x, t))$ is the singled valued set composed only by a couple $(Q(x), 0)$ where Q is a steady solution of the first equation.

Ground states

In what follows we intend a ground state to be a map $Q \in H^1\left(\mathbb{R}^3, \mathcal{S}_0^{(3)}\right)$ such that $(Q(x), 0)$ is a steady solution of the Beris-Edward system and such that $Q(x)$ minimizes the Landau-de Gennes energy among other solutions, i.e. $Q(x)$ is a solution of the following equation:

$$\Delta Q = aQ - b\left(Q^2 - \frac{\operatorname{tr}\left(Q^2\right)}{3} Id\right) + c \operatorname{tr}\left(Q^2\right) Q \tag{3.1}$$

such that that if $W \in H^1\left(\mathbb{R}^3, \mathcal{S}_0^{(3)}\right)$ is another solution of (3.1) then $E(Q) \leq E(W)$.

The constrained problem

In view of the analysis of the general case, i.e. the existence of ground states as defined above where all possible Q -tensors in $\mathcal{S}_0^{(3)}$ are considered, in this work we study a simplified problem where we consider only Q -tensors of a particular form.
Let us consider the family D_0 of Q-tensors with the following structure:

$$Q_q(x) = \begin{bmatrix} -q(x) & 0 & 0 \\ 0 & -q(x) & 0 \\ 0 & 0 & 2q(x) \end{bmatrix}$$

where $q(x) \in H^1\left(\mathbb{R}^3\right)$ is such that $q \geq 0$ a.e. x

The constrained case

If we plug the expression for $(Q_q, 0)$ into the Beris-Edward system, after some straightforward calculations and using the fact that $|q| = q$ a.e. x by definition of D_0, we see that the system boils down to

$$\begin{cases} -\Delta q = -aq + bq|q| - 6cq^3 \\ \nabla p = 6\nabla \cdot (\nabla q \otimes \nabla q) \end{cases}$$

where $(\nabla q \otimes \nabla q)_{i,j} = \partial_i q \partial_j q$.
We point out that the term $Q\Delta Q - \Delta QQ$ in the second equation vanishes due to the diagonal structure of the family D_0.

The constrained energy

Let us consider the natural energy associated to the first equation of the system:

$$S(q) = \frac{1}{2}\int |\nabla q|^2 + \frac{a}{2}\int |q|^2 - \frac{b}{3}\int |q|^3 + \frac{3}{2}c\int |q|^4$$

It holds that:

$$E\left(Q_q\right) = 6S(q),$$

where $E(Q_q)$ is the Landau-de Genes energy.

The constrained case

The previous observation suggests the fact that the problem of minimization of the Landau-de Gennes energy over the Q -tensor in D_0 can be recast in terms of ground states of the equation:

$$-\Delta q = -aq + bq|q| - 6cq^3 \tag{3.2}$$

Definition

A Ground state for equation (3.2) is a solution $q \in H^1(\mathbb{R}^3)$ such that $q \geq 0, q \neq 0$, and for every other $w \in H^1(\mathbb{R}^3)$ that solves equation (3.2) we have

$$S(q) \leq S(w)$$

Constrained Ground states

We are then led to give the following

Definition

A constrained ground state with respect to D_0 for the Beris-Edward equations is a map $q \in H^1\left(\mathbb{R}^3\right), q \geq 0, q \neq 0$ such that Q_q is a weak solution of the system

$$\Delta Q_q = aQ_q - b\left(Q_q^2 - \frac{\mathrm{tr}\left(Q_q^2\right)}{3} Id\right) + c\ \mathrm{tr}\left(Q_q^2\right) Q_q \tag{3.3}$$

and such that $(Q_q, 0)$ is a solution of the Beris-Edward system, furthermore if $w \in H^1\left(\mathbb{R}^3\right)$ is $w \geq 0$ and Q_w is another solution of (3.3) then it must be $E\left(Q_q\right) \leq E\left(Q_w\right)$.

An existence result

The following result guarantees the existence of constrained ground state.

Theorem

There exists a constrained ground state with respect to D_0 for the Beris-Edward system.

Linearization around Ground states

Let $Q_0 \in H^1\left(\mathbb{R}^3, \mathbb{R}^{3\times 3}\right)$ be a ground state for the Beris-Edward system.

We are interested in outlining an analysis of the stability of this ground state with respect to small perturbation, hence we set $Q(t,x) := Q_0(x) + W(t,x)$ where $W \in H^1\left(\mathbb{R}^3, \mathbb{R}^{3\times 3}_{Sym}\right)$ is a small error term, and similarly we take a small perturbation of the fluid velocity: $u = 0 + u$.

The equation for the error term

In order to derive the equations for the error term (W, u) we plug the expression for (Q, u) into the Beris-Edward system.
From the first equation of the system we obtain:

$$\partial_t (Q_0 + W) + u \cdot \nabla (Q_0 + W) = \Omega (Q_0 + W) - (Q_0 + W) \Omega + H_{(Q_0+W)},$$

where $H_{(Q_0+W)}$ is the term H of the original equation with Q replaced by $Q_0 + W$.

The equation for the error terms

If we expand the previous expression we obtain

$$\partial_t W + u \cdot \nabla W = \Omega W - W\Omega + H_W - f\left(Q_0, W\right) - g\left(u, Q_0\right),$$

where

$$\begin{aligned} f(Q,W) = & -\frac{2b}{3}(Q:W)Id - bQW - bWQ, \\ & + c|Q|^2 W + cQ|W|^2 + 2c(Q:W)Q + 2c(Q:W)W \\ g(u,Q) = & \ u \cdot \nabla Q - \Omega Q + Q\Omega. \end{aligned}$$

The equation for the error term

Now we carry out the same substitution in the second equation of the system obtaining: (4.3)

$$\partial_t u + u \cdot \nabla u = \Delta u + \nabla p - \nabla \cdot \left(\nabla \left(Q_0 + W\right) \odot \nabla \left(Q_0 + W\right)\right)$$

$$+\nabla \cdot \left(\left(Q_0 + W\right) \Delta \left(Q_0 + W\right) - \Delta \left(Q_0 + W\right) \left(Q_0 + W\right)\right)$$

The equation for the error term

The last equation might be rewritten in a more intelligible form as follows:

$$\partial_t u + u \cdot \nabla u = \Delta u + \nabla p - \nabla \cdot (\nabla W \odot \nabla W) + \nabla \cdot (W \Delta W - \Delta W W) \\ + h\,(Q_0, W) + z\,(Q_0)\,,$$

which again is the second equation of the system with the two additional terms:

$$\begin{aligned} h(Q, W) &= -\nabla \cdot (\nabla Q \odot \nabla W + \nabla W \odot \nabla Q) \\ &\quad + \nabla \cdot (Q \Delta W + W \Delta Q - \Delta W Q - \Delta Q W), \\ z(Q) &= -\nabla \cdot (\nabla Q \odot \nabla Q) + \nabla \cdot (Q \Delta Q - \Delta Q Q) \end{aligned}$$

The constrained case

In this last part we use the analysis carried out in the previous section adapting it to the D_0 -constrained case for which we actually proved the existence of ground states. The first aspect to be considered is the kind of perturbations that we are concerned with. Indeed, given the special structure of the constrained ground states, we cannot expect them to be stable under all kind of perturbations.

The constrained case

We expect that a non diagonal error W would not lead to stability. On the other hand diagonal matrices of the form (2.3.2), i.e. Q_q, will be shown to be good candidates. To explain why this is the case we first need to analyse the Beris-Edward system in the constrained case. In the constrained setting, the system simplifies considerably.

Constrained stress tensors

If we consider $Q_q \in D_0$ then we have a nice structure for the tensor terms in the second equation since:

$$\nabla Q_q \odot \nabla Q_q = 6 \nabla q \otimes \nabla q,$$

$$Q_q \Delta Q_q = \begin{bmatrix} q\Delta q & 0 & 0 \\ 0 & q\Delta q & 0 \\ 0 & 0 & 4q\Delta q \end{bmatrix},$$

The constrained system

Thus if we plug the expression for such a matrix into the system we obtain the following set of equations:

$$\begin{cases} \partial_t q + u \cdot \nabla q = \Delta q - aq + bq^2 - 6cq^3, & i = j \\ \frac{1}{2}\left(\partial_j u_i - \partial_i u_j\right)\left[\left(Q_q\right)_{jj} - \left(Q_q\right)_{ii}\right] = 0, & i \neq j \\ \partial_t u + u \cdot \nabla u = \Delta u + \nabla p - 6\nabla \cdot (\nabla q \otimes \nabla q) \end{cases}$$

with $\nabla \cdot u = 0$

The constrained system

We immediately see that in general we must have

$$\frac{1}{2}\left(\partial_j u_i - \partial_i u_j\right)\left[\left(Q_q\right)_{jj} - \left(Q_q\right)_{ii}\right] = 0$$

to have a solution of the system of the form Q_q.
There are two possibilities: we could ask the flux to be irrotational, or we can simply consider the case with $u = 0$. We analysed the second case.

Diagonal structure preservation

Proposition

Let $0 < T \leq \infty$ and $q \in C\left(0, T; H^4\right) \cap C^1\left(0, T; H^2\right)$ be a mild solution of the equation

$$\partial_t q = \Delta q - aq + bq^2 - 6cq^3$$

such that $q(0, \cdot)$ is radial, then $(Q_q, 0)$ is a solution of Beris-Edward system.

Being the diagonal structure preserved for $u = 0$ in view of Proposition (4.1.1), it is natural to ask whether the constrained ground states are stable for small perturbation of the form $(Q_{w_0}, 0)$ where $w_0(x)$ is a radial function regular enough.

Equation for the error

Let $(Q_{q_*}, 0)$ be a constrained ground state for the Beris-Edward system in the sense (2.3.9) , and let Q_{w_0} be a small perturbation. We look for solution of the form $(Q_q, 0)$ of the Beris-Edward system with initial data $(Q_{q_*} + Q_{w_0}, 0) = (Q_{q_*+w_0}, 0)$. We want to derive the equations that w must satisfy in order to have that $(Q_q, 0)$ is a solution of the Beris-Edward system.

Equation for the error

we plug into the system the term $(Q_q, 0)$, with $q = q_* + w$, to obtain: (4.10)

$$\begin{cases} \partial_t w = \Delta w - aw + bw^2 - 6cw^3 + 2bq_* w - 18cq_*^2 w - 18cq * w^2 \\ \nabla p = 6\nabla \cdot \left[\nabla \left(q_* + w\right) \otimes \nabla \left(q_* + w\right)\right] \end{cases}$$

Equation for the error

Proposition

Let $w \in C\left(0, T; H^4\right) \cap C^1\left(0, T; H^2\right)$ be a solution of the initial value problem

$$\partial_t w = \Delta w - aw + bw^2 - 6cw^3 + 2bq_* w - 18cq_*^2 w - 18cq_* w^2,$$
$$w(0, x) = w_0(x)$$

with $(t, x) \in (0, T) \times \mathbb{R}^3$, where $w_0 \in H^4\left(\mathbb{R}^3\right)$ is radial. Then $(Q_{q_*+w}, 0)$ is a solution of the Beris-Edward system with initial data $(Q_{q_*+w_0}, 0)$.

Heat equation with potential

We consider the Cauchy problem for a semilinear heat equation with a potential,

$$\begin{cases} \partial_t u = \Delta u - V(x)u + u^p & \text{in } \mathrm{R}^N \times (0,\infty) \\ u(x,0) = \phi(x) \geqslant 0 & \text{in } \mathrm{R}^N \end{cases}$$

where $p > 1, N \geqslant 2, \partial_t = \partial/\partial t, \phi \in L^\infty\left(\mathrm{R}^N\right) \cap C\left(\mathrm{R}^N\right)$, and the potential V is nonnegative and behaves like $\omega|x|^{-2}(1+o(1))$ with $\omega > 0$, as $|x| \to \infty$.

Heat equation with potential

In 1966, Fujita [3] considered the Cauchy problem

$$\begin{cases} \partial_t u = \Delta u + u^p & \text{in } \mathrm{R}^N \times (0, \infty) \\ u(x,0) = \phi(x) \geqslant 0 & \text{in } \mathrm{R}^N \end{cases}$$

and proved that
(A) if $1 < p < p_*$, the problem has no positive global solutions;
(B) if $p > p_*$, the problem has a positive global solution for some initial data ϕ, where $p_* = 1 + 2/N$. We call this critical number $p_* = 1 + 2/N$ the Fujita exponent.

Some results due to Fujita

The statement (A) also holds for the case $p = p_*$, which was proved by Hayakawa [4], Kobayashi, Sirao, and Tanaka [7] and alternative proofs were given by Aronson and Weinberger [1] and Weissler [11]. Subsequently the Fujita result has been extended by many mathematicians in several directions. For the details, see two survey papers [2] and [8] to this problem and references therein.

The work of Zhang

The potential V has a strong influence on the Fujita exponent. Zhang [10] considered the problem of the existence and nonexistence of global positive solutions for the Cauchy problem (1.1) on an $N(\geqslant 3)$-dimensional complete noncompact Riemannian manifold M.

Some results due to Zhang

He studied the relation between the Fujita exponent and the potentials V behaving like $\omega / \left(1 + d(x)^a\right)$, by using global bounds for the fundamental solutions of the heat equations with a potential. Here $\omega \in \mathrm{R}, a > 0$, and $d(x)$ is the distance between a point $x \in M$ and a reference point $O \in M$.

In particular, for the case $M = R^N$ with $N \geqslant 3$ and $a \neq 2$, he proved that the Fujita exponent p_* for the Cauchy problem (1.1) is $1 + 2/N$ if $a > 2, \infty$ if $1 < a < 2$ and $\omega < 0$, and 1 if $1 < a < 2$ and $\omega > 0$. Furthermore, for the case $a = 2$, he also proved that $1 < p_* \leqslant 1 + 2/N$ if $\omega > 0$ and $p_* \geqslant 1 + 2/N$ if $\omega < 0$. In particular, the case $a = 2$ is a border line case where the Fujita exponent may vary from 1 to ∞, and it would be interesting to study the relation between the Fujita exponent p_* and the constant ω.

Some results due to Zhang

There exist positive constants ω, θ, and R such that

$$V(x) \geqslant \omega |x|^{-2} \left(1 - |x|^{-\theta}\right)$$

for all $x \in \mathrm{R}^N$ with $|x| \geqslant R$. Then the Cauchy problem has a global positive solution for some initial data ϕ if

$$p > p_*(\omega)$$

Decay estimates for the sign changing potential

The decay estimates for the heat equation

$$\begin{cases} v_t - \Delta v = 0 \ \text{ in } \ (0,\infty) \times \mathbb{R}^3, \\ v(0,x) = \phi(x). \end{cases} \tag{5.1}$$

are summarized in the following.

Lemma

For any $p \in [2,\infty]$ we have

$$\|v(t)\|_{L^p(\mathbb{R}^3)} \leq Ct^{-3/2+3/p} \|\phi\|_{L^{p'}(\mathbb{R}^3)}. \tag{5.2}$$

Decay estimates for the sign changing potential

this section we study the heat equation with a potential V, i.e.

$$\left| \begin{array}{l} v_t + Lv = 0 \ \text{ in } \ (0,\infty) \times \mathbb{R}^3, \\ v(0,x) = \phi(x). \end{array} \right. \tag{5.3}$$

where $L = -\Delta + V$, the potential $V(x)$ can be decomposed in positive and negative part

$$V(x) = V_+(x) + V_-(x), \quad V_+(x) \geq 0, \ V_1(x) \leq 0.$$

Decay estimates for the sign changing potential

Typically L is assumed to be a positive operator, i.e. there exists $\alpha \in (0, 1/4)$ so that

$$V_-(x) > -\frac{\alpha}{1+|x|^2}. \tag{5.4}$$

Then Theorem 1.1 of
Q. S. Zhang, Global bounds of Schrödinger heat kernels with negative potentials, J. Funct. Anal., 182 (2001),
implies that for any $N > 1$ there exists a constant C_N so that for $0 < t \le N$ and $p \in [2, \infty]$ we have

$$\|e^{-Lt}\phi\|_{L^p(\mathbb{R}^3)} \le C_N t^{-3/2+3/p} \|\phi\|_{L^{p'}(\mathbb{R}^3)}. \tag{5.5}$$

More precisely, the heat kernel

$$e^{-Lt}(x,y)$$

satisfies the estimate

$$\left| e^{-Lt}(x,y) \right| \lesssim t^{-3/2} e^{-c|x-y|^2/t}, \quad 0 < t \leq 1. \tag{5.6}$$

Therefore we can take $p, q \in [1, \infty]$ so that

$$\frac{1}{q} < 1 + \frac{1}{p}$$

and for $t \in (0, N)$ we get

$$\| e^{-Lt} \phi \|_{L^p(\mathbb{R}^3)} \leq C_N t^{-3/(2q)+3/(2p)} \| \phi \|_{L^q(\mathbb{R}^3)} \tag{5.7}$$

We shall assume that $V_\pm$ are small and decay sufficiently rapidly at infinity

$$V_\pm(x) \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3), \|V_\pm\|_{L^1 \cap L^\infty} \leq \delta. \tag{5.8}$$

Our main goal in this section is to extend this result for $t > N$.

The idea can be explained for the simpler case $p = \infty, q = 1$ and $V_+ = 0$.
We start with the representation

$$e^{-Lt}\phi = e^{\Delta t}\phi + \int_0^t e^{\Delta(t-s)} V e^{-Ls}\phi ds. \tag{5.9}$$

Taking $t > 3N$ we decompose the integral as follows

$$\int_0^t e^{\Delta(t-s)} V e^{-Ls}\phi ds = I_1(t) + I_2(t) + I_3(t),$$

where

$$I_1(t) = \int_0^N e^{\Delta(t-s)} V e^{-Ls} \phi ds,$$

$$I_2(t) = \int_N^{t-N} e^{\Delta(t-s)} V e^{-Ls} \phi ds,$$

$$I_3(t) = \int_N^{t-N} e^{\Delta(t-s)} V e^{-Ls} \phi ds.$$

Set

$$\phi(t) = t^{3/2} \|e^{-Lt}\phi\|_{L^\infty}.$$

Then from (5.9) and the decay estimate (5.2) for the free heat equation we have

$$\phi(t) \lesssim \|\phi\|_{L^1} + t^{3/2}\|I_1(t)\|_{L^\infty} + t^{3/2}\|I_2(t)\|_{L^\infty} + t^{3/2}\|I_3(t)\|_{L^\infty}.$$

For $I_1(t)$ we use again (5.2), (5.9) and find

$$|I_1(t)| \lesssim \int_0^N (t-s)^{-3/2}\|Ve^{-Ls}\phi\|_{L^1}ds \lesssim \delta t^{-3/2}\int_0^N \|e^{-Ls}\phi\|_{L^2}ds \lesssim$$

$$\lesssim \delta t^{-3/2}\int_0^N s^{-3/4}\|\phi\|_{L^1}ds \lesssim t^{-3/2}\|\phi\|_{L^1}.$$

For $I_3(t)$ we use (5.2) and find

$$|I_3(t)| \lesssim \int_{t-N}^{t} \|Ve^{-Ls}\phi\|_{L^\infty} ds \lesssim \delta \int_{t-N}^{t} \|e^{-Ls}\phi\|_{L^\infty} ds \lesssim$$

$$\lesssim \delta t^{-3/2} \int_{t-N}^{t} \varphi(s) ds.$$

Finally, we have

$$|I_2(t)| \lesssim \int_{N}^{t-N3} (t-s)^{-3/2} \|Ve^{-Ls}\phi\|_{L^1} ds \lesssim \delta \int_{N}^{t-N} (t-s)^{-3/2} \|e^{-Ls}\phi\|_{L^\infty}$$

$$\lesssim \delta \int_{N}^{t-N} (t-s)^{-3/2} s^{-3/2} \varphi(s) ds.$$

Lemma

If the potential V satisfies (5.8) , *then*

$$\|e^{-Lt}\phi\|_{L^p(\mathbb{R}^3)} \leq C_N t^{-3/(2q)+3/(2p)} \|\phi\|_{L^q(\mathbb{R}^3)} \tag{5.10}$$