EXACT SOLUTIONS OF A GENERALIZED SAWADA – KOTERA EQUATION MODELING NONLINEAR LONG WAVES IN SEVERAL NATURAL PHENOMENA

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Abstract

In this study we discuss existence of solitary wave solutions of a famous higher-order model evolution equation arising from the water wave theory. This evolution equation describes several nonlinear natural phenomena such as propagation of long waves in shallow water or long stratified internal waves in the atmosphere. We obtain new exact analytical solutions of this equation by applying a particular case of the Simplest Equations Method (SEsM). Numerical simulations of the obtained solutions include various kinds of solitary waves depending on key model parameters.

Problem Formulation

In this study we consider the generalized Sawara – Kotera equation, presented in the form:

$$(1) \qquad \frac{\partial u}{\partial t} + \alpha u^2 \frac{\partial u}{\partial x} + \beta u \frac{\partial^3 u}{\partial x^3} + \delta \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^5 u}{\partial x^5} = 0,$$

which describes several nonlinear natural phenomena such as propagation of long waves in shallow water or long stratified internal waves in the atmosphere.

We shall search for an analytical solution of this equation applying a particular variant of the Simple Equations Method (SEsM) [N. K. Vitanov (2019) The simple equations method (SEsM) for obtaining exact solutions of nonlinear PDEs: Opportunities connected to the exponential functions. *AIP Conference Proceedings* 2159, 030038], i.e. so called Modified Method of Simplest Equation (MMSE), which is developed for the case when we use one simple equation and the solution is searched as power series of the solution of the simple equation [N. K. VITANOV (2011) On Modified Method of Simplest Equation for Obtaining Exact and Approximate Solutions of Nonlinear PDEs: the Role of the Simplest Equation. *Communications in Nonlinear Science and Numerical Simulation* **16** 4215-4231].

Methodology

- First of all, by means of an appropriate ansatz (for an example the traveling-wave ansatz) the solved nonlinear partial differential equation is reduced to a nonlinear ordinary differential equation of a kind:

$$P(u_\xi, u_{\xi\xi}, u_{\xi\xi}.....) = 0 \qquad (2)$$

- Then the solution $u(\xi)$ is searched as some function of another function $f(\xi)$.

  Often this function is a finite-series solution:

$$u(\xi) = \sum_{\mu=-\nu}^{\nu_1} a_m [f(\xi)]^m \qquad (3)$$

  where a_m are coefficients and $f(\xi)$ is solution of simpler ordinary differential equation called simplest equation.

- Eq. (3) is substituted in Eq. (2) and let the result of this substitution be a polynomial of $f(\xi)$. Then a balance procedure is applied. It has to ensure that all the coefficients of the obtained polynomial of $f(\xi)$ contain more than one term.

Methodology

- This procedure leads to one or more balance equations relating some of the parameters of the solved equation and some of the parameters of the solution. Eq. (3) is a candidate for solution of Eq. (2) if all coefficients of the obtained polynomial of $f(\xi)$ are equal to 0.

This condition leads to a system of nonlinear algebraic equations for the coefficients of the solved nonlinear PDE and for the coefficients of the solution. Any nontrivial solution of this algebraic system leads to a solution the studied nonlinear partial differential equation.

- We shall consider traveling-wave $u(x,t) = u(\xi) = u(\mu x + vt)$ constructed on the basis solutions of the simplest equation

$$f_\xi^2 = n^2 (f^2 - f^{\frac{2n+2}{n}}) \qquad (4)$$

where n is arbitrary positive real number. The solution of this equation is $f(\xi) = \dfrac{1}{\cosh^n(\xi)}$

- The balance procedure leads to the balance equations *m=2/n* and *m=1/n*. For our further investigation we shall use the balance equation *m=2/n*.

SEVERAL SOLUTIONS OF THE GENERALIZED SAWARA – KOTERA EQUATION

Acording to the MMSE methodology we introduce the traveling –wave coordinate

$$u(x,t) = u(\xi) = u(\mu x + vt)$$

in Eq. (1) and we search for a solution of the type (3), as we use Eq. (4) as simplest equation.

We shall consider two particular cases connected to the value *n* presented in the simplest equation (4).

I. <u>CASE n = 1</u>

In this case Eq. (4) becomes

$$f_{\xi}^{2} = f^{2} - f^{4} \qquad (5)$$

and its solution is

$$f(\xi) = \frac{1}{\cosh(\xi)}. \qquad (6)$$

In accordance with the balance equation *m=2/n*, the solution of Eq. (1) takes the form:

$$u(\xi)=a_0+a_1 f(\xi)+a_2 f(\xi)^2 \tag{7}$$

The relationships among coefficients of the solution and coefficients of the model are obtained by solving a system of six algebraic equations, and they are:

$$a_0=-20\frac{\beta}{10\alpha-\delta^2},\ \ a_1=0,\ \ a_2=3\frac{\beta(10\alpha+\delta^2+\sqrt{10\alpha-\delta^2})}{10\alpha-\delta^2},\ \mu=\frac{\beta\delta}{10\alpha-\delta^2}, \tag{8}$$

$$v=-16\frac{\beta^5\delta^3(25\beta-4\delta^2)}{100000\alpha^5-50000\alpha^4\delta^2+1000\alpha^3\delta^4+1000\alpha^2\delta^6+50\alpha\delta^8-\delta^{10}}$$

Then the solution of Eq. (1) can be presented as:

$$u(\xi)=-20\frac{\beta}{10\alpha-\delta^2}+3\frac{\beta(10\alpha+\delta^2+\sqrt{10\alpha-\delta^2})}{10\alpha-\delta^2}f(\xi) \tag{9}$$

where $f(\xi)$ is determined by Eq. (6) and α,β,δ are free parameters.

II. CASE n = 2

In this case Eq. (4) becomes

$$f_{\xi}^{2} = 4(f^{2} - f^{3}) \qquad (10)$$

and its solution is

$$f(\xi) = \frac{1}{\cosh^{2}(\xi)} \qquad (11)$$

In accordance with the balance equation *m=2/n*, the solution of Eq. (1) takes the form:

$$u(\xi) = a_0 + a_1 f(\xi) \qquad (12)$$

The relationships among coefficients of the solution and coefficients of the model are obtained by solving a system of six algebraic equations, and they are:

$$(13)\quad a_0 = -\frac{16}{5}\frac{\delta^2\beta}{\alpha^2}, \quad a_1 = \frac{48}{5}\frac{\delta^2\beta}{\alpha^2}, \quad \mu = \frac{1}{5}\frac{\delta\ \beta}{\alpha}, \quad v = \frac{256}{3125}\frac{\delta^5\beta^5}{\alpha^5}$$

Then the solution of Eq. (1) can be presented as:

$$(14)\qquad u(\xi) = -\frac{16}{5}\frac{\delta^2\beta}{\alpha^2} + \frac{48}{5}\frac{\delta^2\beta}{\alpha^2} f(\xi)$$

where $f(\xi)$ is determined by Eq. (11) and α, β, δ are free parameters.

Several Numerical Illustrations of the Obtained Solutions

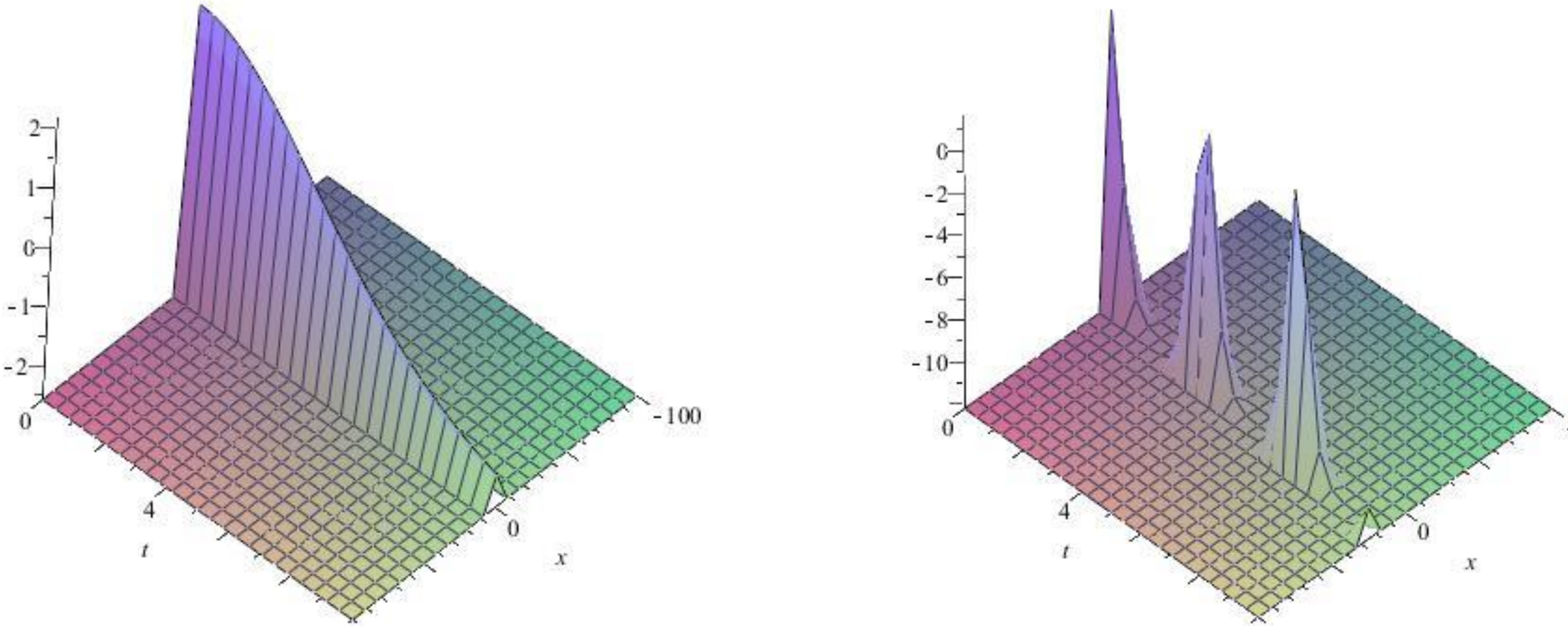

Fig. 1. Solitons connected to the solution (11) of Eq. (1) for $\alpha = 0.5, \delta = 0.05$ and $\beta = 0.05$ (left), $\beta = 0.5$ (right)

CONCLUDING REMARKS

In this study we obtain two exact traveling wave solutions of the Sawara - Kotera equation based on the simplest equation for the function $1/\cosh^{n}(\mu x+vt)$.

We present only a particular numerical picture of the possible solitary waves which can be derived on the basis of one of the obtained solutions and varying in value only of one model parameter. A wider analysis of the obtained solution can be derived by means of variation of larger number of parameters. The obtained traveling wave solutions and their numerical illustations can be useful to test for tsunamis predictions, atmospheric flows, storm surges, flows around structures, etc All these topics will be discused in our further investigations.

Thank you for your attention !