High Order Symplectic Finite Difference Scheme for Double Dispersion Equations

V. Vucheva, N. Kolkovska

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences, Sofia, Bulgaria

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Introduction

We consider the Cauchy problem for the Boussinesq equation

$$\frac{\partial^2 U}{\partial t^2} = \Delta U + \beta_1 \Delta \frac{\partial^2 U}{\partial t^2} - \beta_2 \Delta^2 U + \Delta f(U), \; x \in \mathbb{R}, \; t > 0, \quad (1)$$

$$U(x,0) = U_0(x), \;\; \frac{\partial U}{\partial t}(x,0) = U_1(x), \quad (2)$$

on the unbounded region $\mathbb{R}$ with asymptotic boundary conditions

$$U(x,t) \to 0, \; \Delta U(x,t) \to 0, \text{ as } |x| \to \infty,$$

where $\Delta = \frac{\partial^2}{\partial x^2}$, the constants $\beta_1 > 0$, $\beta_2 > 0$ are dispertion parameters and α is an amplitude parameter.

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$$f(U) = \alpha U^p, p = 2, 3, ..., p \in N$$

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Mathematical Models

- describing the spreading of longitudinal strain wave in an isotropic compressible elastic rod
  A. Porubov, Amplification of nonlinear strain waves in solids, World Scientific,2003;
- in the propagation a small amplitude wave on the surface of shallow water
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References

- C. Christov, Conservative difference scheme for Boussinesq model of surface waves,1996
- N. Kolkovska, M. Dimova, D. Vassileva, K. Angelow.

Hamiltonian System

Canonical form

$$\begin{bmatrix} \dfrac{\partial U}{\partial t} \\ \dfrac{\partial V}{\partial t} \end{bmatrix} = \mathrm{J} \begin{bmatrix} \dfrac{\delta H}{\delta U} \\ \dfrac{\delta H}{\delta V} \end{bmatrix}$$

$$\frac{\partial V}{\partial x} = \frac{\partial U}{\partial t}, \quad \mathrm{J} = \begin{bmatrix} 0 & (I_d - \beta_1 \Delta)^{-1} \partial_x \\ (I_d - \beta_1 \Delta)^{-1} \partial_x & 0 \end{bmatrix}$$

Hamiltonian

$$H(U,V) = \frac{1}{2} \int_R \left(V^2 + U^2 + \beta_1 \left(\frac{\partial V}{\partial x} \right)^2 + \beta_2 \left(\frac{\partial U}{\partial x} \right)^2 + \frac{2\alpha U^{p+1}}{p+1} \right) dx$$

Hamiltonian system

$$\frac{\partial U}{\partial t} = \frac{\partial V}{\partial x},$$
$$\frac{\partial V}{\partial t} = (I_d - \beta_1 \Delta)^{-1} \left(\frac{\partial U}{\partial x} - \beta_2 \frac{\partial^3 U}{\partial x^3} + \alpha \frac{\partial U^p}{\partial x} \right).$$

Symplecticness

The solving mapping $\Phi_H(t_0, t)$, defined by

$$(U, V) = \Phi_H(t_0, t)(U_0, V_0)$$

is an area-preserving transformation.

Checking preservation of the area

Φ_H preserves the area and orientation $\Longleftrightarrow$ ${\psi'}^T J \psi' = J$, where ψ' is the Jacobian matrix of Φ_H.

Theorem (Conservation laws)

For every $t \geq 0$ the solution $U(x,t)$ to the DDE satisfies the following identities

- $H(U(x,t)) = H(U(x,0))$, *i.e. conservation of the Hamiltonian (energy)*
  $H(U,V) = \frac{1}{2} \int_{\mathbb{R}} \left(V^2 + U^2 + \beta_1 \left(\frac{\partial V}{\partial x}\right)^2 + \beta_2 \left(\frac{\partial U}{\partial x}\right)^2 + \frac{2\alpha U^{p+1}}{p+1} \right) dx;$
- $I(U(x,t)) = I(U(x,0))$, *i.e. conservation of the mass*
  $I(U(x,t)) = \int_{\mathbb{R}} U(x,t) dx;$
- $M(U(x,t), V(x,t)) = M(U(x,0), V(x,0))$, *i.e. conservation of the momentum*
  $M(U(x,t), V(x,t)) = \int_{\mathbb{R}} \left(U(x,t)V(x,t) + \beta_1 U_x(x,t) V_x(x,t) \right) dx.$

Notation

- Domain $\Omega = [-L_1, L_2]$ with an uniform grid $x_0, ..., x_N$ with a step h;
- $u_i(t) \approx U(x_i, t), \ v_i(t) \approx V(x_i, t)$;
  $\vec{u}(t) = (u_1(t), ..., u_N(t)), \ \vec{v}(t) = (v_1(t), ..., v_N(t))$;
- notations for some discrete derivatives of the mesh functions:

$$\begin{aligned}
\partial_{\hat{x}} u_i &:= u_{\hat{x},i} = \frac{u_{i+1} - u_{i-1}}{2h}, \\
\hat{\Delta} u &:= u_{\hat{x}\hat{x},i} = \frac{u_{i+2} - 2u_i + u_{i-2}}{4h^2}, \\
u_{\hat{x}\hat{x}\hat{x},i} &:= \frac{u_{i+3} - 3u_{i+1} + 3u_{i-1} - u_{i-3}}{8h^3}.
\end{aligned}$$

Semi-discrete Hamiltonian

$$H_h(\vec{u}(t), \vec{v}(t)) = \frac{1}{2} \sum_{i=1}^{N} h \left(v_i^2 + \beta_1 v_{\hat{x},i}^2 + u_i^2 + \beta_2 u_{\hat{x},i}^2 + \frac{2\alpha}{p+1} u_i^{p+1} \right)$$

Semi-discrete Hamiltonian

$$H_h(\vec{u}(t), \vec{v}(t)) = \frac{1}{2} \sum_{i=1}^{N} h \left(v_i^2 + \beta_1 v_{\hat{x},i}^2 + u_i^2 + \beta_2 u_{\hat{x},i}^2 + \frac{2\alpha}{p+1} u_i^{p+1} \right)$$

After evaluation of the variational derivatives of H_h with respect to unknowns $u_i(t)$ and $v_i(t)$, $i = 1, 2, ..., N$ we get

Semi-discrete system of $2N$ ODE

$$\frac{du_i(t)}{dt} = v_{\hat{x},i},$$
$$\frac{dv_i(t)}{dt} = (I_d - \beta_1 \hat{\Delta}_h)^{-1} \left(u_{\hat{x},i} - \beta_2 u_{\hat{x}\hat{x}\hat{x},i} + \alpha \left(u^p\right)_{\hat{x},i} \right)$$

$$A = \frac{1}{2h}\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & -1 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix}, B = \frac{1}{4h^2}\begin{bmatrix} 2 & 0 & -1 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 2 \end{bmatrix}$$

$$f(\vec{u}) = \alpha(u_1^p, u_2^p, ..., u_N^p)^T$$

Separable Hamiltonian System

$$\frac{d\vec{u}}{dt} = A\vec{v}$$

$$(I_d + \beta_1 B)\frac{d\vec{v}}{dt} = A(\vec{u} + \beta_2 B\vec{u} + f(\vec{u}))$$

The matrix $J_h = (I_d + \beta_1 B)^{-1} \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ satisfies the requirements for Poisson brackets from Hairer, Lubich and Wanner, *Geometric numerical integration*, 2004 and generates the Poisson brackets

The 3-stage Lobatto IIIA and IIIB methods

0	0	0	0
1/2	5/24	1/3	−1/24
1	1/6	2/3	1/6
	1/6	2/3	1/6

a) Lobatto IIIA

0	1/6	-1/6	0
1/2	1/6	1/3	0
1	1/6	5/6	0
	1/6	2/3	1/6

b) Lobatto IIIB

$$\vec{u}^{n+1} = \vec{u}^n + \frac{\tau}{6}\left(\vec{k_1} + 4\vec{k_2} + \vec{k_3}\right), \quad \vec{v}^{n+1} = \vec{v}^n + \frac{\tau}{6}\left(\vec{l_1} + 4\vec{l_2} + \vec{l_3}\right),$$

$$\begin{aligned}
\vec{k_1} &= A\vec{v}^n, \;\; \vec{k_2} = A\left(\vec{v}^n + \frac{\tau}{24}(5\vec{l_1} + 8\vec{l_2} - \vec{l_3})\right), \;\; \vec{k_3} = A\left(\vec{v}^n + \frac{\tau}{6}(\vec{l_1} + 4\vec{l_2} + \vec{l_3})\right) \\
\vec{l_1} &= (I_d + \beta_1 B)^{-1} A\left\{\left(\vec{u}^n + \frac{\tau}{6}(\vec{k_1} - \vec{k_2})\right) + \right. \\
&+ \left. \beta_2 B\left(\vec{u}^n + \frac{\tau}{6}(\vec{k_1} - \vec{k_2})\right) + \alpha\left(\vec{u}^n + \frac{\tau}{6}(\vec{k_1} - \vec{k_2})\right)^p\right\}, \\
\vec{l_2} &= (I_d + \beta_1 B)^{-1} A\left\{\left(\vec{u}^n + \frac{\tau}{6}(\vec{k_1} + 2\vec{k_2})\right) + \right. \\
&+ \left. \beta_2 B\left(\vec{u}^n + \frac{\tau}{6}(\vec{k_1} + 2\vec{k_2})\right) + \alpha\left(\vec{u}^n + \frac{\tau}{6}(\vec{k_1} + 2\vec{k_2})\right)^p\right\}, \\
\vec{l_3} &= (I_d + \beta_1 B)^{-1} A\left\{\left(\vec{u}^n + \frac{\tau}{6}(\vec{k_1} + 5\vec{k_2})\right) + \right. \\
&+ \left. \beta_2 B\left(\vec{u}^n + \frac{\tau}{6}(\vec{k_1} + 5\vec{k_2})\right) + \alpha\left(\vec{u}^n + \frac{\tau}{6}(\vec{k_1} + 5\vec{k_2})\right)^p\right\}.
\end{aligned}$$

Properties

The approximation error of the time integration method is $O(\tau^4)$, thus the overall error of the discrete method is $O(h^2 + \tau^4)$.

Theorem

For every n=0,...,K the discrete FDS conserves exactly

- *the discrete symplectic structure* $\omega^n = d\vec{z}^n \wedge J_h d\vec{z}^{n-1}$, *where* $\vec{z}^n = (\vec{u}^n, \vec{v}^n)$ *(the scheme is symplectic);*
- *the discrete mass* $I_h\left(t^n\right) := \sum_{i=1}^{N} hu_i^n$.

Algorithm

A. Evaluate $u^{(0)}$, $u^{(1)}$ from the initial conditions

B. For $n = 1, 2, \ldots, K$

1. The stage vector $\vec{k}_1$ is computed by $\vec{v}$ at nth time layer
2. $\vec{k}_1^{[0]} = \vec{k}_2^{[0]} = \vec{k}_3^{[0]} = A\vec{v}^n$
3. for $s = 1, 2, \ldots$ continue until the iterations converge, i.e. until $\max\{|\vec{l}_i^{[s]} - \vec{l}_i^{[s+1]}|, |\vec{k}_i^{[s]} - \vec{k}_i^{[s+1]}|\} < tol, \quad i = 1, 2, 3,$ where tol is a prescribed precision.
4. $$\vec{u}^{n+1} = \vec{u}^n + \frac{\tau}{6}\left(\vec{k}_1^{[s+1]} + 4\vec{k}_2^{[s+1]} + \vec{k}_3^{[s+1]}\right),$$
$$\vec{v}^{n+1} = \vec{v}^n + \frac{\tau}{6}\left(\vec{l}_1^{[s+1]} + 4\vec{l}_2^{[s+1]} + \vec{l}_3^{[s+1]}\right).$$

Propagation of a solitary wave

Exact solution of the 1D equation

$$\tilde{u}(x,t;c) = \frac{3(c^2-1)}{2\alpha}\operatorname{sech}^2\left(-\frac{1}{2}\sqrt{\frac{c^2-1}{\beta_1 c^2-\beta_2}}(x-ct)\right)$$

Error: $\psi_h = max_{\,0\le k\le N}\|\tilde{u}_h^k - u_h^k\|$

Order of convergence: $\kappa = \log_2 \dfrac{\|\tilde{u}_h^k - u_h^k\|}{\|\tilde{u}_{h/2}^k - u_{h/2}^k\|}$

Runge's rule

Propagation of a solitary wave

$$x \in [-80, 80], \;\; T = 20, \;\; \alpha = 3, \;\; \beta_1 = 1.5, \;\; \beta_2 = 0.5, \;\; c = 2.$$

$$u(x,0) = \tilde{u}(x,0;c), \qquad v(x,0) = -c\tilde{u}(x,0;c)$$

Table: Rate of convergence κ with respect to h ($\tau = 0.5h$) is $O(h^2)$

h	τ	Error ψ_h	Rate κ	Max iter.	Time (sec.)
0.2	0.1	$2.0 * 10^{-2}$		7	1
0.1	0.05	$5.0 * 10^{-3}$	1.9923	6	4
0.05	0.025	$1.2 * 10^{-3}$	1.9988	5	12
0.025	0.0125	$3.1 * 10^{-4}$	1.9996	5	47
0.0125	0.00625	$7.8 * 10^{-5}$	1.9999	5	186

Propagation of a solitary wave

$$x \in [-80, 80],\ \ T = 20,\ \ \alpha = 3,\ \ \beta_1 = 1.5,\ \ \beta_2 = 0.5,\ \ c = 2.$$

H_h^K- discrete energy; H-exact energy

Table: Rate of convergence of the energy κ_E for Problem 1

h	τ	H_h^K	$\lvert H - H_h^K \rvert$	Rate κ_E
0.2	0.1	32.924380896	$1.5 * 10^{-2}$	
0.1	0.05	32.935546089	$3.7 * 10^{-3}$	1.9956
0.05	0.025	32.938348048	$9.3 * 10^{-4}$	1.9989
0.025	0.0125	32.939049207	$2.3 * 10^{-4}$	1.9997
0.0125	0.00625	32.939224539	$5.8 * 10^{-5}$	1.9999

In all examples the relative mass error is of order 10^{-15}.

Propagation of solitary wave

Table: Rate of convergence κ with respect to τ ($h = 2\tau^2$)

τ	h	Error ψ_h	Rate κ	Max.iter	Time (sec.)
0.4	0.32	0.0294879273		20	0
0.2	0.08	0.0018559663	3.9899	8	1
0.1	0.02	0.0001160519	3.9993	16	7

Rate of convergence of the discrete solution to the exact one is $O(\tau^4)$

Interaction of two solitary waves

$x \in [-160, 160]$, $T = 50$, $\alpha = 3$, $\beta_1 = 1.5$, $\beta_2 = 0.5$,
$c_1 = 1.1$, $c_2 = -1.3$

Initial condition:
$u(x,0) = \tilde{u}\,(x + 30, 0; c_1) + \tilde{u}\,(x - 40, 0; c_2)$
$v(x,0) = -c_1 \tilde{u}\,(x + 30, 0; c_1) - c_2 \tilde{u}\,(x - 40, 0; c_2)$

Error:

$$\psi_{h/4} = \frac{\left\| u_{[h]} - u_{[h/2]} \right\|^2}{\left\| u_{[h]} - u_{[h/2]} \right\| - \left\| u_{[h/2]} - u_{[h/4]} \right\|},$$

Order:

$$\kappa = \log_2 \left(\frac{\left\| u_{[h]} - u_{[h/2]} \right\|}{\left\| u_{[h/2]} - u_{[h/4]} \right\|} \right)$$

Interaction of two solitary waves

$x \in [-160, 160]$, $T = 80$, $\alpha = 3$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $c_1 = 1.1$, $c_2 = -1.3$

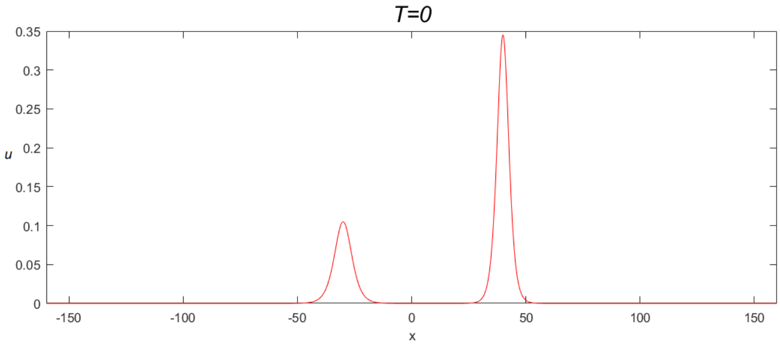
Table: Rate of convergence κ with respect to h ($\tau = 0.5h$) is $O(h^2)$

h	τ	Error $\psi_{h/4}^K$	Rate κ	Max iter.	Time(sec)
0.2	0.1			6	4
0.1	0.05			6	18
0.05	0.025	0.004474784	1.9886	5	58
0.025	0.0125	0.001125320	1.9972	4	186
0.0125	0.00625	0.000281743	1.9992	4	900

Rate of convergence of the discrete solution to the exact one with respect to τ ($h = 2\tau^2$) is $O(\tau^4)$

Collision of two waves with $c_1 = 1.1$ and $c_2 = -1.3$

$$x \in [-160, 160],\ T = 80,\ \alpha = 3,\ \beta_1 = 1.5,\ \beta_2 = 0.5,\ p = 2$$



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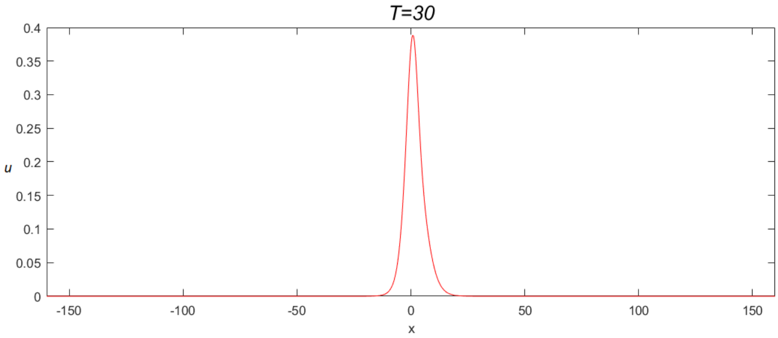

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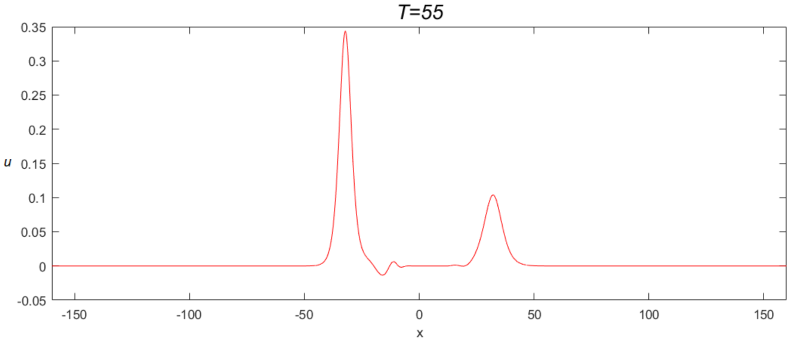

Conclusion

- In this paper a numerical method with approximation order $O(h^2 + \tau^4)$ is applied for the solution of the double dispersion equation.
- The method preserves exactly the symplectic structure of the discrete solution and the discrete mass.
- The reported numerical experiments show convergence of the discrete solution to the exact one with second order with respect to space step and fourth order with respect to time step, for single solitary wave and two colliding solitary waves.

Thank you for your attention!