Adomian's decomposition method and homotopy perturbation method in solving two-dimensional Volterra-Fredholm fuzzy integral equations

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Introduction

The study of fuzzy Volterra-Fredholm integral equations begins in

1. Kaleva O., Fuzzy differential equations, Fuzzy Sets and Systems, 1987, Vol.24, P.301–317.

2. Seikkala S., On the fuzzy initial value problem, Fuzzy Sets and Systems, 1987, Vol.24, P.319–330.

3. Mordeson J., Newman W., Fuzzy integral equations, Information Sciences, 1995, Vol.87, P.215-–229.

Introduction

The homotopy perturbation method proposed by He for solving linear and nonlinear integral equations.

4. Allahviranloo T., Hashemzehi S.: The homotopy perturbation method for fuzzy Fredholm integral equations.: Journal of Applied Mathematics. **5** 1–12 (2008)

proposed homotopy perturbation method to solve fuzzy Fredholm integral equations, respectively

5. Allahviranloo T., Khezerloo M., Ghanbari M. and Khezerloo S.: The homotopy perturbation method for fuzzy Volterra integral equations. International Journal of computional cognition. **12** 31–37 (2010)

proposed method to solve fuzzy Volterra integral equations.

Introduction

Adomian decomposition method (ADM) has been recently intensively studied by scientists and engineers and used for solving nonlinear differential and integral problems. The ADM introduced by Adomian in

6. G. Adomian, A review of the decomposition method in applied mathematics, Journal of Mathematical Analysis and Applications, 1988, Vol.135, P.501–544.

7. G. Adomian, Solving Frontier Problems of Physics: The Decompositon Method, Kluver Academic Publishers, Boston, 1994.

for solving different kind of functional equations and has been subject of extensive numerical and analytical studies.

Introduction

In this work we investigate the nonlinear 2D-VFFIE

$$u(s,t) = g(s,t) \oplus (FR)\int_a^b k_1(s,t,x) \odot G_1(u(x,t))dx \oplus$$
$$\oplus (FR)\int_c^t k_2(s,t,y) \odot G_2(u(s,y))dy \oplus (FR)\int_c^t\int_a^b k_3(s,t,x,y) \odot G_3(u(x,y))dxdy,$$

where $g,\ u : A = [a,b] \times [c,d] \to E^1$ are continuous fuzzy-number valued functions, $k_1 : A \times [a,b] \to R_+, k_2 : A \times [c,d] \to R_+$, $k_3 : A \times A \to R_+$ and $G_1,\ G_2,\ G_3 : E^1 \to E^1$ are continuous functions. The set E^1 is the set of all fuzzy numbers.
The aim of this paper is to show that the HPM with a specific convex homotopy for solving nonlinear 2D-VFFIE is equivalent to the ADM.

Preliminaries

Definition

A fuzzy number is a function $u : \mathbb{R} \to [0, 1]$ satisfying the following properties:
(i) u is upper semi-continuous.
(ii) $u(x) = 0$ outside of some interval $[c, d]$.
(iii) there are the real number a and b with $c \leq a \leq b \leq d$,such that u is increasing on $[c, a]$, decreasing on $[b, d]$ and $u(x) = 1$ for each $x \in [a, b]$.

The set of all fuzzy numbers is denoted by E^1. Any real number $a \in \mathbb{R}$ can be interpreted as a fuzzy number $\tilde{a} = \chi_{[a]}$ and therefore $\mathbb{R} \subset E^1$.

Preliminaries

For any $0 < r \leq 1$ we denote the $r-$level set $[u]^r = \{x \in \mathbb{R} : u(x) \geq r\}$ that is a closed interval and $[u]^r = [u^r_-, u^r_+]$ for all $r \in [0, 1]$.
These lead to the usual parametric representation of a fuzzy number: $[u]^r = [u^r_-, u^r_+]$ for all $r \in [0, 1]$, where u_-, u_+ can be considered as functions $u_-, u_+ : [0, 1] \to \mathbb{R}$, such that u_- is increasing and u_+ is decreasing.

Preliminaries

For $u, v \in E^1$, $k \in \mathbb{R}$, the addition and the scalar multiplication are defined by $[u+v]^r = [u]^r + [v]^r = [u_-^r + v_-^r, u_+^r + v_+^r]$ for all $r \in [0,1]$

$$[k.u]^r = k.[u]^r = \begin{cases} [ku_-^r, ku_+^r], & \text{if } k \geq 0 \\ [ku_+^r, ku_-^r], & \text{if } k < 0 \end{cases}$$

As a distance between fuzzy numbers we use the Hausdorff metric defined by

$D(u,v) = \sup_{r \in [0,1]} max(|u_-^r - v_-^r|, |u_+^r - v_+^r|)$ for any $u, v \in E^1$

Preliminaries

Lemma

The Hausdorff metric has the following properties:
(i) (E^1, D) *is a complete metric space,*
(ii) $D(u \oplus w, v \oplus w) = D(u, v)$ *for all* $u, v, w \in E^1$,
(iii) $D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e)$ *for all* $u, v, w, e \in E^1$,
(iv) $D(u \oplus v, \tilde{0}) \leq D(u, \tilde{0}) + D(v, \tilde{0})$ *for all* $u, v \in E^1$,
(v) $D(k \odot u, k \odot v) = |k| D(u, v)$ *for all* $u, v \in E^1$, *for all* $k \in \mathbb{R}$.

For any fuzzy-number-valued function $f : A \to E^1$ we can define the functions $\underline{f}(.,.,r)$, $\overline{f}(.,.,r) : A \to R$, by $\underline{f}(s,t,r) = (f(s,t))^r_-$, $\overline{f}(s,t,r) = (f(s,t))^r_+$ for each $(s,t) \in A$, for each $r \in [0,1]$. These functions are called the left and right $r-$level functions of f.

Preliminaries

Definition

A fuzzy-number-valued function $f : A \to E^1$ *is said to be continuous at* $(s_0, t_0) \in A$ *if for each* $\varepsilon > 0$ *there is* $\delta > 0$ *such that* $D(f(s,t), f(s_0,t_0)) < \varepsilon$ *whenever* $((s-s_0)^2 + (t-t_0)^2)^{\frac{1}{2}} < \delta$. *If* f *be continuous for each* $(s,t) \in A$ *then we say that* f *is continuous on* A.

Lemma

If $f : A \to E^1$ *is continuous then it is bounded and its supremum* $\sup\limits_{(s,t)\in A} f(s,t)$ *must exist and is determined by* $u \in E^1$ *with* $u_-^r = \sup\limits_{(s,t)\in A} \underline{f}(s,t,r)$ *and* $u_+^r = \sup\limits_{(s,t)\in A} \overline{f}(s,t,r)$. *A similar conclusion for the infimum is also true.*

Parametric form of 2D-VFFIE

In this section we introduce the parametric form of nonlinear 2D-VFFIE.
Let $(\underline{u}(s,t,r), \overline{u}(s,t,r))$ and $(\underline{g}(s,t,r), \overline{g}(s,t,r))$, $0 \leq r \leq 1$ and $(s,t)] \in A$ be parametric form of the function $u(s,t)$ and $g(s,t)$, respectively.
We consider the case when the functions $G_1(\beta)$, $G_2(\beta)$ and $G_3(\beta)$ are increasing for $\beta \in [\underline{u}(x,y,r), \overline{u}(x,y,r)]$ and $k_1(s,t,x) \geq 0$, $k_2(s,t,y) \geq 0$, $k_3(s,t,x,y) \geq 0$. Then the parametric form of the nonlinear 2D-VFFIE is,

Parametric form of 2D-VFFIE

$$\overline{u}(s,t,r) = \overline{g}(s,t,r) + \int_a^b k_1(s,t,x) G_1(\overline{u}(x,t,r))dx + \\ + \int_c^t k_2(s,t,y) G_2(\overline{u}(s,y,r))dy + \int_c^t \int_a^b k_3(s,t,x,y) G_3(\overline{u}(x,y,r))dxdy,$$

$$\underline{u}(s,t,r) = \underline{g}(s,t,r) + \int_a^b k_1(s,t,x) G_1(\underline{u}(x,t,r))dx + \\ + \int_c^t k_2(s,t,y) G_2(\underline{u}(s,y,r))dy + \int_c^t \int_a^b k_3(s,t,x,y) G_3(\underline{u}(x,y,r))dxdy,$$

Now, we explain ADM as a numerical algorithm for approximating solution of this system of nonlinear integral equations in crisp case. Then, we find approximate solution for $u(s,t,r)$

ADM for 2D-VFFIE

The ADM assume an infinite series solution for the unknowns functions $(\underline{u}(s,t,r), \overline{u}(s,t,r))$, given by

$$\underline{u}(s,t,r) = \sum_{i=0}^{\infty} \underline{u_i}(s,t,r), \quad \overline{u}(s,t,r) = \sum_{i=0}^{\infty} \overline{u_i}(s,t,r)$$

The nonlinear operators $G_1(\underline{u})$, $G_1(\overline{u})$, $G_2(\underline{u})$, $G_2(\overline{u})$, $G_3(\underline{u})$, $G_3(\overline{u})$ into an infinite series of polynomials given by

$$G_1(\underline{u}) = \sum_{n=0}^{\infty} \underline{A_n}(\underline{u}_0, \underline{u}_1, \ldots, \underline{u}_n), \quad G_1(\overline{u}) = \sum_{n=0}^{\infty} \overline{A_n}(\overline{u}_0, \overline{u}_1, \ldots, \overline{u}_n),$$
$$G_2(\underline{u}) = \sum_{n=0}^{\infty} \underline{B_n}(\underline{u}_0, \underline{u}_1, \ldots, \underline{u}_n), \quad G_2(\overline{u}) = \sum_{n=0}^{\infty} \overline{B_n}(\overline{u}_0, \overline{u}_1, \ldots, \overline{u}_n),$$
$$G_3(\underline{u}) = \sum_{n=0}^{\infty} \underline{C_n}(\underline{u}_0, \underline{u}_1, \ldots, \underline{u}_n), \quad G_3(\overline{u}) = \sum_{n=0}^{\infty} \overline{C_n}(\overline{u}_0, \overline{u}_1, \ldots, \overline{u}_n).$$

ADM for 2D-VFFIE

Adomian polynomial $A_n = (\underline{A}_n, \overline{A}_n)$, $B_n = (\underline{B}_n, \overline{B}_n)$, $C_n = (\underline{C}_n, \overline{C}_n)$ $n \geq 0$ defined by

$$\underline{A}_n(\underline{u}_0, \underline{u}_1, \ldots, \underline{u}_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[G_1\left(\sum_{i=0}^{\infty}\lambda^i \underline{u}_i\right)\right]_{\lambda=0},$$

$$\overline{A}_n(\overline{u}_0, \overline{u}_1, \ldots, \overline{u}_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[G_1\left(\sum_{i=0}^{\infty}\lambda^i \overline{u}_i\right)\right]_{\lambda=0},$$

$$\underline{B}_n(\underline{u}_0, \underline{u}_1, \ldots, \underline{u}_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[G_2\left(\sum_{i=0}^{\infty}\lambda^i \underline{u}_i\right)\right]_{\lambda=0},$$

$$\overline{B}_n(\overline{u}_0, \overline{u}_1, \ldots, \overline{u}_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[G_2\left(\sum_{i=0}^{\infty}\lambda^i \overline{u}_i\right)\right]_{\lambda=0},$$

$$\underline{C}_n(\underline{u}_0, \underline{u}_1, \ldots, \underline{u}_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[G_3\left(\sum_{i=0}^{\infty}\lambda^i \underline{u}_i\right)\right]_{\lambda=0},$$

$$\overline{C}_n(\overline{u}_0, \overline{u}_1, \ldots, \overline{u}_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[G_3\left(\sum_{i=0}^{\infty}\lambda^i \overline{u}_i\right)\right]_{\lambda=0},$$

where λ is formal parameter.

ADM for 2D-VFFIE

Then we get

$$\sum_{n=0}^{\infty} \underline{u}_n(s,t,r) = \underline{g}(s,t,r) + \int_a^b k_1(s,t,x) \sum_{n=0}^{\infty} \underline{A}_n dx + \\ + \int_c^t k_2(s,t,y) \sum_{n=0}^{\infty} \underline{B}_n dy + \int_c^t \int_a^b k_3(s,t,x,y) \sum_{n=0}^{\infty} \underline{C}_n dxdy$$

$$\sum_{n=0}^{\infty} \overline{u}_n(s,t,r) = \overline{g}(s,t,r) + \int_a^b k_1(s,t,x) \sum_{n=0}^{\infty} \overline{A}_n dx + \\ + \int_c^t k_2(s,t,y) \sum_{n=0}^{\infty} \overline{B}_n dy + \int_c^t \int_a^b k_3(s,t,x,y) \sum_{n=0}^{\infty} \overline{C}_n dxdy$$

ADM for 2D-VFFIE

The components $\underline{u}_n(s,t,r)$ and $\overline{u}_n(s,t,r)$, $n \geq 0$ are computed using the following recursive relations

$$\underline{u}_0(s,t,r) = \underline{g}(s,t,r)$$

$$\underline{u}_1(s,t,r) = \int_a^b k_1(s,t,x)\underline{A}_0 dx + \int_c^t k_2(s,t,y)\underline{B}_0 dy +$$

$$+ \int_c^t \int_a^b k_3(s,t,x,y)\underline{C}_0 dxdy$$

$$\ldots$$

$$\underline{u}_n(s,t,r) = \int_a^b k_1(s,t,x)\underline{A}_n dx + \int_c^t k_2(s,t,y)\underline{B}_n dy +$$

$$+ \int_c^t \int_a^b k_3(s,t,x,y)\underline{C}_n dxdy$$

ADM for 2D-VFFIE

$$\overline{u}_0(s,t,r) = \overline{g}(s,t,r)$$

$$\overline{u}_1(s,t,r) = \int_a^b k_1(s,t,x)\overline{A}_0 dx + \int_c^t k_2(s,t,y)\overline{B}_0 dy +$$

$$+ \int_c^t \int_a^b k_3(s,t,x,y)\overline{C}_0 dxdy$$

$$\ldots$$

$$\overline{u}_n(s,t,r) = \int_a^b k_1(s,t,x)\overline{A}_n dx + \int_c^t k_2(s,t,y)\overline{B}_n dy +$$

$$+ \int_c^t \int_a^b k_3(s,t,x,y)\overline{C}_n dxdy$$

HMP for 2D-VFFIE

To explain the HPM, we consider the linear operator

$$L(u(s,t)) = (L(\underline{u}(s,t,r)), L(\overline{u}(s,t,r))$$

where

$$L(\underline{u}(s,t,r)) = \underline{u}(s,t,r) - \underline{g}(s,t,r),$$
$$L(\overline{u}(s,t,r) = \overline{u}(s,t,r) - \overline{g}(s,t,r)$$

and nonlinear operator

$$N(u(s,t)) = (N(\underline{u}(s,t,r)), N(\overline{u}(s,t,r)))$$

where

HMP for 2D-VFFIE

$$N(\underline{u}(s,t,r)) = \underline{u}(s,t,r) - \underline{g}(s,t,r) - \int_a^b k_1(s,t,x)G_1(\underline{u}(x,t,r))dx -$$
$$- \int_c^t k_2(s,t,y)G_2(\underline{u}(s,y,r)))dy - \int_c^t \int_a^b k_3(s,t,x,y)G_3(\underline{u}(x,y,r))dxdy,$$

$$N(\overline{u}(s,t,r)) = \overline{u}(s,t,r) - \overline{g}(s,t,r) - \int_a^b k_1(s,t,x)G_1(\overline{u}(x,t,r))dx -$$
$$- \int_c^t k_2(s,t,y)G_2(\overline{u}(s,y,r)))dy - \int_c^t \int_a^b k_3(s,t,x,y)G_3(\overline{u}(x,y,r))dxdy,$$

HPM for 2D-VFFIE

Classically, we choose a convex homotopy by

$$H(u(s,t),p) = (H(\underline{u}(s,t,r),p), H(\overline{u}(s,t,r),p)$$

$$H(\underline{u}(s,t,r),p) = (1-p)L(\underline{u}(s,t,r)) + pN(\underline{u}(s,t,r)),$$
$$H(\overline{u}(s,t,r),p) = (1-p)L(\overline{u}(s,t,r)) + pN(\overline{u}(s,t,r)),$$

with properties

$$H(u(s,t),0) = L(u(s,t)),\ H(u(s,t),1) = N(u(s,t)),$$

where $p \in [0,1]$ is an embedding parameter.

HPM for 2D-VFFIE

The embedding parameter p monotonically increases from zero to unit as trivial problem

$$H(u(s,t),0),$$

is continuously deformed to original problem

$$H(u(s,t),1),$$

where $u(s,t)$ is a solution of nonlinear 2D-VFFIE.

HPM for 2D-VFFIE

The HPM uses the homotopy parameter p as an expanding parameter to obtain

$$\underline{U}(\underline{u}(s,t,r),p) = \sum_{i=0}^{\infty} p^i \underline{u_i}(s,t,r),$$
$$\overline{U}(\overline{u}(s,t,r),p) = \sum_{i=0}^{\infty} p^i \overline{u_i}(s,t,r),$$

where $p \to 1$ becomes the approximate solution of nonlinear 2D-VFFIE i.e.

$$\underline{u}(s,t,r) = \lim_{p\to 1} \underline{U}(\underline{u}(s,t,r),p),$$
$$\overline{u}(s,t,r) = \lim_{p\to 1} \overline{U}(\overline{u}(s,t,r),p).$$

Equivalence HPM and ADM

In this section, we investigate the equivalence of the convergent HPM and the convergent ADM for the solution of nonlinear 2D-VFFIE. We show that the HPM is equivalent to the ADM with a specific convex homotopy and vice versa.

Theorem

The homotopy perturbation method is equivalent to Adomian's decomposetion method, for nonlinear 2D-VFFIE, with the homotopy $H(u(s,t),p)$ given by
$H(u(s,t),p) = (1-p)L(u(s,t)) \oplus pN(u(s,t)) = 0,$

Thank you
for your attention!