Semilinear Riemann- Liouville evolution inclusions with causal operators

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Let $(E, |\cdot|)$ be a real Banach space and let $A : D(A) \to E$ be a densely defined linear operator. In this talk we study the following nonlocal fractional evolution inclusion

$$\begin{cases} ({}^{L}D^{q}_{0+}x)(t) = Ax(t) + f_x(t), \quad t \in [0, b] = I \\ f_x(t) \in F(t, (Qx)(t)), \\ |s|^{1-q}x(s) = g(x(\cdot))(s), \quad s \in [-\tau, 0], \end{cases} \tag{1}$$

Here $({}^{L}D^{q}_{0+}\cdot)$ is the Riemann-Liouville fractional derivative of order q, $0 < q < 1$, with the lower limit zero, I^{1-q}_{0+} is Riemann-Liouville integral of order $1-q$ and Q is a causal operator. Furthermore, $x_0(0) = \lim\limits_{s\to 0} |s|^{1-q}x_0(s)$ assuming that the limit exists.

Evolution inclusion (1) w.r.t. Caputo derivative is studied in the literature under some compactness or dissipative type assumptions. The **dissipative assumptions are Lipschitz conditions.** When $F(t, \cdot)$ is Lipschitz, then one uses Picar successive approximations and can prove the existence of solutions and some properties of the solution set under appropriate assumptions. For R-L derivative problem we refer to

Z. Mey, J. Peng, Existence of mild solution for Riemann–Liouville fractional differential equations with nonlocal conditions, *ArXiv: 1507.08540vl* (2015).

The compactness type assumptions are mainly in two directions.
(I) Assume that A generates compact semigroup. In this case the state space is separable and one has to use some conditions of the image of $F(\cdot,\cdot)$ has weakly compact values or the Banach space Y is assumed to be reflexive.
(II) Assume that A generates an equicontinuous evolution semigroup while $F(\cdot,\cdot)$ satisfies condition with respect to some measure of noncompactness. In this case the solution set of (1) is nonempty and compact. We use this approach.

Y. Zhou, L. Ziag, X. Shen, Existence of mild solutions for fractional evolution equations, *J. Int. Eqns Appl.* **25)** (2013) 557-586.

We refer the reader also to:

Burlica M., Necula M., Rosu D., Vrabie I., *Delay Differential Evolutions Subject to Nonlocal Initial Conditions* CRC Press Taylor & Francis group, New York, 2016.

Zhou Y., *Fractional Evolution Equations and Inclusions: Analysis and Control.* Academic Press, 2016.

S. Zhu, Z. Fan, I. Lu, Characterizations of solution set for fractional evolution equations and applications to control systems, *Top. Meth. Nonlin. Anal.* DOI 2775yTMNA2019033.

Denote $\bar{I} = [-\tau, b]$ and let $I' = (0, b]$.

$$C_q(\bar{I}, E) = \{x \in C(\bar{I} \setminus \{0\}], E) : \lim_{t \to 0} |t|^{1-q} x(t) \text{ exists}\}.$$

The norm of this apace is given by $\|z\|_q = \sup_{t \in \bar{I}} |t|^{1-q} |z(t)|$. We define

$$z_q(t) = \begin{cases} |t|^{1-q} z(t) & t \in \bar{I} \setminus \{0\} \\ \lim_{t \to 0} |t|^{1-q} z(t) & t = 0. \end{cases}$$

Analogously we defite $C_q(I, E)$. For $\Omega \subset C_q(I, X)$, define Ω_q by

$$\Omega_q = \{z_q(\cdot) : \; z(\cdot) \in \Omega\}, \tag{2}$$

Lemma 1.

A set $\Omega \subset C_q(I, E)$ is relatively compact if and only if Ω_q is relatively compact in $C(I', E)$.

Definition 2.

An operator $Q : C_{1-q}(\bar{I}, X) \to L^p(I, X)$ is called causal if for every $s \in (I'$ and any $u(\cdot), v(\cdot) \in C_{1-q}(\bar{I}, X)$ such that $u(t) = v(t)$ when $t \in [-\tau, s]$ one has $(Qu)(t) = (Qv)(t)$ for $t \in [0, s]$.

Definition 2 implies that the casual operator is similar to delay operator x_t, however $(Qu)(t)$ depends on the values of $u(\cdot)$ in the whole interval $[-\tau, t]$ while $x_t(\cdot)$ which depends on the values of x only on $[t - \tau, t]$. Of course $x_t(\cdot)$ is a particular case of casual operator.

If $x \in L^p(I,E)$, then

$$I^q x(t) := \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} x(s)ds, \tag{3}$$

exists. for $t \in I$ - Reimann-Liouville fractional integral of order $q > 0$. Γ is Gamma function. Let $0 < q < 1$. If $x \in L^p(I,E)$ is such that $t \mapsto I^{1-\alpha} u(t)$ is differentiable a.e. on I, then

$$({}^L D^q_{0^+} x)(t) := \frac{d}{dt} \left(I^{1-q} x\right)(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} x(s)ds. \tag{4}$$

and exists a.e. on I - Riemann-Liouville fractional derivative of order q with initial point 0. .

Let $\phi_q(t) : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi_q(t) = \begin{cases} \frac{t^{1-q}}{\Gamma(q)}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

Then

$$I^q x(t) = (\phi_q * x)(t) \text{ and } D^q x(t) = \frac{d}{dt}(\phi_{1-q} * x)(t).$$

Lemma 3.

Let $q, p \in \mathbb{R}_+$. *Then*

$$\int_0^1 t^{q-1}(1-t)^{p-1}dt = \frac{\Gamma(q)\Gamma(p)}{\Gamma(q+p)}$$

and hence

$$\int_0^s t^{q-1}(x-t)^{p-1}dt = s^{q+p-1}\frac{\Gamma(q)\Gamma(p)}{\Gamma(q+p)}.$$

The integral in the first equation of Lemma 3 is known as Beta function $\beta(q,p)$.

Let A generate a C_0- semigroup $P(\cdot)$.

Definition 4.

The continuous function $x(\cdot)$ is said to be a solution of (1) with if

$$x(t) = t^{q-1} S_q(t)x(0) + \int_0^t (t-s)^{q-1} S_q(t-s) f_x(s) ds, \; t \in I',$$

Furthermore, $x(0) = \lim_{s \to 0} |s|^{1-q} x(s)$ assuming that the limit exists. where $f_x(s) \in F(s, x(s))$ is strongly measurable.

Here

$$S_q(t) = q\int_0^{\infty} \theta\xi(q)P(t^q\theta)d\theta,$$

$$\xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\bar{\omega_q}(\theta^{-\frac{1}{q}}) \text{ and}$$

$$\bar{\omega}(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-nq-1}\frac{\Gamma(nq+1)}{n!}sin(n\pi q), \theta \in (0,\infty),$$

$$\xi_q(\theta) \geq 0, \int_0^{\infty}\xi_q(\theta)d\theta = 1, \int_0^{\infty}\theta\xi_q(\theta)d\theta = \frac{1}{\Gamma(1+q)}, \theta \in (0,+\infty).$$

Hausdorff MNC defined by:

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon - \text{net}\}.$$

We will use also sequential MNC, generated by $\chi(\cdot)$:

$$\chi_0(B) = \sup\{\chi(\{x_n : \; n \geq 1\}) : \; (x_n) \text{ is a sequence in } B\}.$$

If X is separable, then $\chi_0(\Omega) = \chi(\Omega)$. In arbitrary X we have

$$\chi_0(\Omega) \leq \chi(\Omega) \leq 2\chi_0(\Omega). \tag{5}$$

for every bounded $\omega \subset X$.

Let $L = \max_{t\in[0,b]} \|P(t)\|$ and $M = \dfrac{L}{\Gamma(q)}$. We assume that:

(A) A is densely defined linear operator, generating an equicontinuous semigroup $P(\cdot)$.

(F1) The operator $Q : C_q(I, X) \rightrightarrows L^p(I, X),\ p > \dfrac{1}{q}$ is continuous, while $F(\cdot, \cdot)$ is almost USC.

(F2) There exist two constants $a,\ e$ such that $\|F(t, (Qy)(t))\| \leq a + e\|(Qy)(t)\|$ for every $y \in E$. There exist $\alpha(\cdot) \in L^p(I, \mathbb{R}_+)$ where $p > \dfrac{1}{q}$ such that

$$\|(Qx)(t)\| \leq a(t),\ \text{for all } x \in E \text{ and for a.e. } t \in (0, b]$$

(F3) There exists $L^p(I, \mathbb{R}_+)$ function $\kappa(\cdot)$ such that for every bounded $B \subset E$ we have

$$\chi(QB)(t) \le \kappa(t)\chi(B),$$

where $p > \frac{1}{q}$. There exists a constant m such that $\chi(F(t, C)) \le m\chi(C)$ for every bounded sets $C \subset E$.

(G) $g : : C_q \to E$ is continuous and compact map such that

$$\|g(x(\cdot))\| \le c\|x_q(\cdot)\| + d, \quad \forall x \in C(I, E),$$

for some positive constants $0 < c < \dfrac{1}{M}$ and $d > 0$.

Theorem 5.

Under the assumptions (A),(F1)–(F3) and (G) the problem (1) *has a solution when* $\frac{M}{\Gamma(q)}\left(c + e\int_0^t (t-s)^{q-1}\alpha(s)ds\right) \le N < 1$. *The solution set is compact.*

Let $Sol(z)$ be the solution set of

$$\begin{cases} ({}^{L}D^{q}_{0^+}x)(t) = Ax(t) + f_x(t), \text{ a.e. } t \in I' = (0, T], \\ f_x(t) \in F(t, t^{1-q}z(t)), \\ x_0 = g(z(\cdot)). \end{cases}$$

where $z(\cdot) \in C_q(I, E)$.
The proof is divided on some steps.

- The solution set of (1) is bounded.
- $Sol(\cdot)$ is contractive with closed graph.
- There exists a convex compact set K such that $Sol : K \rightrightarrows K$ is with non empty compact values.

Afterward use

Theorem 6.

Let Θ be a convex compact subset of a Banach space. If $\Psi : \Theta \rightrightarrows \Theta$ is with closed graph and compact contractive values, then there exists a fixed point $z \in \Psi(z)$.

Clearly every fixed point of *Sol* is a solution of (1).
The solution set is closed and due to (F3) also compact.

Example.

Let Ω be a nonempty bounded open set in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Denote $X = L^p(\Omega)$, with $1 \leq p < \infty$ and $a \in \mathbb{R}^n$. Let $\vec{l} = (l_1, l_2 ..., l_n)$ is a fixed vector with $\sum_{i=1}^{n} |l_i| = \frac{1}{\Gamma(q)}$,

$$\nabla_x u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right).$$

$D_t^q u$ is the partial fractional derivative of u w.r.t. t in Riemann-Liouville sense.
We define the operator $A : D(A) \subset E \to E$ by

$$\begin{cases} D(A) = \{w \in X, \ \vec{l}\nabla w \in X\} \\ Aw = \vec{k}\nabla w. \end{cases}$$

Then A is a densely defined linear operator, generating a semigroup $T(\cdot)$ given by $T(t)u = u(x + t\vec{l})$. Clearly $T(\cdot)$ is equicontinuous (it is isometry), but not compact. Moreover in our case $L = 1$, i.e. $M = \frac{1}{\Gamma(q)}$.

Consider the fractional partial (transport) differential inclusion with finite delay

$$\begin{cases} ({}^L D^q_{0+} u_t)(t,x) \in Au + G(t, \tilde{u}_t(x)), & [0,b] \times \Omega, \\ u(t,x) = 0, & (0,b] \times \partial\Omega \\ |\theta|^{1-q} u(\theta,x) = \int_{-h}^{\theta} \left(\int_{\Omega} \Phi(s,x,\lambda,|s|^{1-q} u(s,\lambda)) d\lambda \right) ds & \theta \in [-h,0], x \in \Omega, \end{cases} \quad (6)$$

where the partial derivatives are taken in the sense of distributions over Ω. Furthermore, $\tilde{u}_t(t,x) = \max_{\tau \in [-h,t]} |\tau|^{t-q} u(\tau,x)$, $(\theta \in [-h,0])$ and

$$G(t, \tilde{u}_t(x)) = [f_1(t, \tilde{u}_t(x)), f_2(t, \tilde{u}_t(x))].$$

$$F(t,u) = \{\lambda \in [0,1] : \ \lambda f_1(t, \tilde{u}_t) + (1-\lambda) f_2(t, \tilde{u}_t)\} \quad (7)$$

is a closed interval for each $(t, \tilde{u}_t) \in [0,b] \times C_{1-q}([-h,0], E)$.

To model (6), we assume that

(A1) $f_i : [0,b] \times C_{1-q}([-h,0], E) \to \mathbb{R}$, $i = 1,2$ are given functions such that $f_1(t, \tilde{u}_t) \leq f_2(t, \tilde{u}_t)$ for each $(t, \tilde{u}_t) \in [0,b] \times C_{1-q}([-h,0], E)$.

(A2) f_1 is lower semicontinuous and f_2 is upper semicontinuous as real valued functions.

(A3) There exists $\eta_1 \in L^\infty(J, \mathbb{R})$ such that

$$|f_i(t,\cdot)| \leq \eta_1(t), \quad i = 1,2. \text{ and } (t, \tilde{u}_t) \in [0,b] \times C_{1-q}([-h,0], E).$$

Denote

$$\begin{cases} D(A) = \{u \in X; \quad a \cdot \nabla u \in X\} \\ Au = a \cdot \nabla u \\ u(t)(x) = u(t,x), \\ f_i(t, u(t,x)) = f_i(t, u(t))(x), \;\; i = 1, 2. \end{cases}$$

Thus A generates a noncompact semigroup $T(t)$ given by

$$T(t)u = u(x - ta), \quad \text{for each } u \in X, \;\; t \in \mathbb{R}.$$

Clearly, the semigroup $T(t)$ is continuous in the uniform operator topology (it is isometry).
(H1 There exist two constants c and d such that $\|\Phi(t,x,\lambda,\alpha)\| \leq c\|\alpha\| + d$. for every $\alpha \in C_q(\Omega)$, where $C_q(\Omega) \supset \|\alpha\| \leq k$.
(H2) For every $k > 0$ there exists a positive function m_k such that

$$\|\Phi(t,x,\lambda,\alpha) - \Phi(t,y,\lambda,\alpha)\| \leq m_k(t,x,y,\lambda)$$

where $\|\alpha\| \leq k$.
(H3) $\lim_{x \to y} \int_{-h}^{a} \left(\int_{\Omega} m_k(t,x,y,\lambda) d\lambda \right) dt = 0$ uniformly on Ω.
(F) There exists $\eta_2 \in L^\infty(J, \mathbb{R})$ such that

$$\chi(F(t,D)) \leq \eta_2(t)\chi(D).$$

Clearly the initial condition is continuous on θ. From Theorem s 4.1 and 4.2 in chapter 5 of

Martin R., *Nonlinear Operators and Differential Equations in Banach Spaces*, Wiley, New York.

for every fixed $.\theta$ it maps bounded sets into relative compacts. Thus condition (G) of Theorem 5 holds. The following theorem is then a corollary of the main result.

Theorem 7.

Under the assumptions (A1)–(A3), (H1), (H2) and (F) the problem (6) *has a solution if*

$$M\left(c + \frac{eb^q}{q}\right) < 1.$$

Thanks for your attention