Local and global space-time integrability for the inhomogeneous heat equation

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*Joint work with Elena Nikolova and George Venkov (TU-Sofia)

Outline

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Consider the Cauchy problem associated to the nonlinear inhomogeneous heat equation (INLH)

$$\begin{cases} u_t - \Delta_x u + k|x|^{-b}|u|^{\alpha}u = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ u(0,x) = f(x) \in H^{\sigma}(\mathbb{R}^d), \end{cases} \tag{1}$$

for $d \geq 1$, where $k \in \mathbb{R}$, u is a real-valued function and the Δ_x is the classical d-dimensional Laplace operator. Moreover $0 < b < \min\{2, d\}$ and the power associated to the nonlinearity α satisfies:

$$\frac{4-2b}{d} < \alpha < \alpha^*_{\sigma}(d), \quad \alpha^*(d) = \begin{cases} \frac{4-2b}{d-2\sigma} & \text{if} \quad d \geq 3 \\ +\infty & \text{if} \quad d = 1, 2. \end{cases} \tag{2}$$

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We denote, for any $\sigma > 0$,

$$H^\sigma(\mathbb{R}^d) = (1-\Delta_x)^{-\frac{\sigma}{2}} L^2(\mathbb{R}^d), \quad \dot{H}^\sigma(\mathbb{R}^d) = (-\Delta_x)^{-\frac{\sigma}{2}} L^2(\mathbb{R}^d)$$

moreover we indicate by $f \in L^q(\mathbb{R}^d)$, for $1 \leq r < \infty$, if

$$\|f\|_{L^r(\mathbb{R}^d)}^r = \|f\|_{L^r_x}^r = \int_{\mathbb{R}^d} |f(x)|^r \, dx < +\infty.$$

We introduce the following further notation: for any Banach space X we define,

$$\|f\|_{L^q_t X} = \left(\int_{\mathbb{R}_+} \|f(x)\|_X^q \, dt \right)^{1/q}.$$

For any two positive real numbers a, b, we write $a \lesssim b$ to denote $a \leq Cb$, with $C > 0$, we unfold the constant only when it is essential.

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The solution u associated to (1) satisfies two conservation laws:

$$\|u(t)\|_{L^2_x} + \int_0^t E(u(s))ds = \|f\|_{L^2_x}, \tag{3}$$

$$\frac{d}{dt}E(u(t)) = -\|u_t(t)\|^2_{L^2_x}, \tag{4}$$

with

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx + \frac{k}{\alpha+2}\int_{\mathbb{R}^d} |x|^{-b}|u(x)|^{\alpha+2} dx.$$

It is possible to investigate some relevant questions as:

- Local and global existence as well as the persistence of regularity for the map data-solution $f \to u(t,\cdot)$, assuming the initial data in is the space $H^\sigma(\mathbb{R}^d)$.
- The long-time behavior of the solutions to (1) in the space $H^\sigma(\mathbb{R}^d)$.

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Motivation arising from Mathematical Physics

- The dissipative quasi-geostrophic (QG) model.
- The Navier-Stokes equation

$$u_t - \Delta_x u - (u \cdot \nabla) u + \nabla P = 0, \quad \nabla \cdot u = 0.$$

- The convection-diffusion equation

$$u_t - \Delta_x u = a \cdot \nabla(|u|^\alpha u), \quad \alpha > 0, \quad a \in \mathbb{R}^d \backslash 0.$$

- The Ginzburg-Landau (GL) equation

$$u_t - (a + ib)\Delta u + F(u, \overline{u}) = 0,$$

with $a, b \geq 0$ being real parameters.

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The INLH (1) (with $b = 0$) has been extensively studied in the past decades by several authors, both in focusing and defocusing setting. For instance, in the context of Lebesgue spaces $L^q(\mathbb{R}^d)$ and imposing certain constraints on the nonlinearity parameter $\alpha > 0$, Weissler (1980, 1981) and Brezis and Cazenave (1996) proved local well-posedness (in the second paper it is showed unconditional uniqueness also). Later on, Haraux and again Weissler (1982) were able to prove also a conditional global well posedness and for the first time the $H^1(\mathbb{R}^d)$-theory was discussed in a seminal way. We would mention also more recent works like the ones of Gazzola and Weth (2005) in which the behavior of solutions to nonlinear heat equations is explored and of Weissler *et al.* (2011), where the stability theory is considered.

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We want to mention also to the work of Ikeda and Taniguchi (2019) (and references therein) in which the global behavior of solutions for the focusing energy-subcritical and critical nonlinear heat equation in general energy spaces are presented in in unified manner. A fundamental tool to study the properties for solutions to the nonlinear problem of type of (1) it is the Stricharz estimates technique. These were obtained for the first time by Strichartz (1977) and were successively developed in various dispersive equations frameworks by a broad number of authors. We recall Keel and Tao (1998), Ginibre and Velo (1989, 1995) concerning the NLS, Georgiev and Visciglia (2003) for the wave equation perturbed by a potential and Georgiev, Stefanov and T. (2006) for the Schrödinger equation perturbed by a magnetic potential.

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There is only a limited literature related to the Strichartz estimates for the heat equation, despite it satisfies properties close to the Schrödinger equations. Let us refer to Miao, B. Yuan, B. Zhang (2006) and Z. Zhai (2009), where Strichartz type estimates for fractional nonlinear heat equations are achieved. We recommend besides the paper by S. Gustafson and D. Roxanas in which the remarkable concentration and compactness technique developed by C. Kënig and F. Merle are adapted to the H^1-critical focusing NLH for $d = 4$, that is (1) with $\alpha = 2$.

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We present here a peculiar approach to display well-posedness in $H^\sigma(\mathbb{R}^d)$ for the solution to (1) (both in focusing and defocusing case) which relies on a combined action of classical and extended Strichartz estimates. This generalized estimates for the heat equation were introduced for the first time by T, E. Nikolova and G. Venkov (2019). We underline that our techniques strongly simplifies the different approaches available in the papers already cited, moreover it guarantees to treat the low space dimensions $d = 1, 2$. In addition it allows to avoid to differentiate at a fractional order the nonlinearity $|x|^{-b}|u|^\alpha u$ with respect to the x variable.

Outline

Our first main outstanding is the following

Theorem 1

Assume $d \geq 1$, $\frac{4-2b}{d} < \alpha < \alpha^*(d)$. Then, there exist an $\frac{\alpha d-(4-2b)}{2\alpha} < \sigma \leq 1$ and (at least) a pair $(q,r) \in \mathbb{R}^2_+$ such that the problem (1) satisfies a unique local solution

$$u(t,x) \in L^q((0,T); L^r(\mathbb{R}^d)), \tag{5}$$

for any initial data $f \in H^\sigma(\mathbb{R}^d)$ and where $T = T(\|f\|_{H^\sigma(\mathbb{R}^d)}) > 0$. The solution can be extended globally if:

- *$k > 0$, that is, equation (1) is defocusing;*
- *$k < 0, |k| << 1$ and $\|f\|_{\dot{H}^{\frac{\alpha d-(4-2b)}{2\alpha}}(\mathbb{R}^d)} < \delta$, with $\delta > 0$.*

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By the Fourier transform and Duhamel's principle, the solution of INLH (1) with a generic forcing term can be written as:

$$u(t,x) = e^{t\Delta_x} f(x) + k \int_0^t e^{(t-\tau)\Delta_x} F(\tau, x)\, d\tau, \tag{6}$$

where $e^{t\Delta_x} = \mathscr{F}^{-1} e^{-t|\xi|^2}$. Since the heat kernel $e^{-t|\xi|^2}$ has a very strong decay, the free propagator $e^{t\Delta_x}$ obeys various decay estimates, for $t > 0$,

$$\left\| e^{t\Delta_x} f \right\|_{L_x^r} \lesssim \|f\|_{L_x^r},\ 1 \le r \le \infty, \tag{7}$$

$$\left\| e^{t\Delta_x} f \right\|_{L_x^\infty} \lesssim t^{-\frac{d}{2r}} \|f\|_{L_x^r},\ 1 \le r \le \infty, \tag{8}$$

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Choosing opportune values for p, r we can achieve the untruncated decay estimate for any $\tau, t \in (0, \infty)$ and such that $\tau \neq t$,

$$\left\| e^{(t\pm\tau)\Delta_x} f \right\|_{L_x^\infty} \lesssim |t \pm \tau|^{-\frac{d}{2}} \|f\|_{L_x^1} \lesssim |t - \tau|^{-\frac{d}{2}} \|f\|_{L_x^1}. \qquad (10)$$

These inequalities lead to the classical homogeneous Strichartz estimate. Although the free propagator is not unitary but self-adjoint operator, (3) and (10) allow us to apply the techniques of Keel and Tao (1998). See also Ginibre and Velo (1995).

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We start by the following:

Definition 1

We say that the pair (q, r) *is* σ**-admissible** *(*$(q, r) \in \mathcal{A}^\sigma$*) if, for* $0 \leq s < \frac{d}{2}$,

$$2 \leq q, r \leq \infty, \quad \frac{2}{q} + \frac{d}{r} = \frac{d}{2} - \sigma, \quad (q, r, d) \neq (2, \infty, 2). \tag{11}$$

Proposition 1 (Homogeneous sharp Strichartz)

Let be $d \geq 1$*. Then the following estimates hold*

$$\|e^{t\Delta_x} f\|_{L_t^q L_x^r} \leq C\|f\|_{H_x^\sigma}, \tag{12}$$

when $(q, r) \in \mathcal{A}^\sigma$ *and satisfy the following*

- $4 \leq q \leq \infty$ *and* $2 \leq r \leq \infty$, *for* $d = 1$;
- $2 < q \leq \infty$ *and* $2 \leq r \leq \infty$, *for* $d = 2$;
- $2 \leq q \leq \infty$ *and* $2 \leq r \leq \infty$, *for* $d \geq 3$.

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Following Foschi (see also Vilela (2007)) we give

Definition 2

We say that the pair (q, r) *is* **acceptable** *if*

$$1 \leq q < \infty, \quad 2 \leq r \leq \infty, \quad \frac{1}{q} < d\left(\frac{1}{2} - \frac{1}{r}\right) \quad or \quad (q, r) = (\infty, 2). \tag{13}$$

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Then by T, Nikolova and Venkov (2019)

Proposition 2 (Inhomogeneous extended non-sharp Strichartz)

Let (q, r) and $(\tilde{q}, \tilde{r})$ be acceptable pairs. Then the following estimate holds for $d \geq 3$

$$\left\| \int_0^t e^{(t-\tau)\Delta_x} F(\tau, x) d\tau \right\|_{L_t^q L_x^r} \leq C \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}, \tag{14}$$

and the conditions

$$\frac{1}{q} + \frac{1}{\tilde{q}} = \frac{d}{2}\left(\frac{1}{\tilde{r}'} - \frac{1}{r}\right), \tag{15}$$

$$\frac{1}{q} < \frac{1}{\tilde{q}'}, \qquad \frac{d-2}{d} \leq \frac{r}{\tilde{r}} \leq \frac{d}{d-2}. \tag{16}$$

Furthermore, if $d = 1$, only condition (15) it is required, while if $d = 2$, is also needed that $r, \tilde{r} < \infty$.

Remark 1

Furthermore, we underline that the two previous lemmas remain valid once one replaces the free heat propagator $e^{t\Delta}$ by $e^{(a+ib)t\Delta}$, with a and b real parameters. This fact will be useful for discussing more general nonlinear equations based on the heat (and Schrödinger) equation.

Remark 1

Furthermore, we underline that the two previous lemmas remain valid once one replaces the free heat propagator $e^{t\Delta}$ *by* $e^{(a+ib)t\Delta}$, *with a and b real parameters. This fact will be useful for discussing more general nonlinear equations based on the heat (and Schrödinger) equation.*

Our first contribution can be stated as follows.

Lemma 1

For any $d \geq 3, k \in \mathbb{R}, 0 < b < \min\{2, d\}$ *and* α *satisfying the assumption* (2) *there exists a real* $s^* = \frac{d\alpha-(4-2b)}{2\alpha} \in (0,1)$, *at least a pair* $(q,r) \in \mathcal{A}^{s^*}$ *and a corresponding acceptable pair* $(\tilde{q},\tilde{r})$ *that fulfils*

$$\frac{1}{\tilde{r}'} = \frac{\alpha+1}{r} + \frac{b}{d}, \quad \frac{1}{\tilde{q}'} = \frac{(\alpha+1)}{q}, \tag{17}$$

such that the solution to the INLH (1) *enjoys both the homogeneous* (12) *and the inhomogeneous extended Strichartz* (14) *estimates.* *Furthermore for* $d = 1, 2$ *we get the same conclusion, provided that we drop the conditions appearing in* (16).

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Sketch of the proof. Fix $\sigma = 1$. We show that to any $d \geq 1$ and α as in (2) corresponds at least one quadruple $(q, r, \tilde{q}, \tilde{r}) \in \mathbb{R}^4_+$ that enables a combined action of the homogeneous Strichartz estimates (12) and the extended inhomogeneous ones (14) by proving that condition (17) and the following inequalities (developed in the previous propositions):

$$0 \leq \frac{1}{r}, \frac{1}{\tilde{r}} \leq \frac{1}{2}, \quad 0 < \frac{1}{q} \leq \frac{1}{2}, \quad 0 < \frac{1}{\tilde{q}} \leq 1, \qquad \text{(a, b, c, d)}$$

$$\frac{1}{q} + \frac{1}{\tilde{q}} < 1, \quad \frac{d-2}{d} \leq \frac{r}{\tilde{r}} \leq \frac{d}{d-2}, \qquad \text{(e, f)}$$

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} - s^*, \quad \frac{1}{\tilde{q}} + \frac{d}{\tilde{r}} < \frac{d}{2}, \qquad \text{(g, h)}$$

$$\frac{1}{q} + \frac{d}{r} < \frac{d}{2}, \quad \frac{2}{q} + \frac{d}{r} + \frac{2}{\tilde{q}} + \frac{d}{\tilde{r}} = d. \qquad \text{(i, j)}$$

are all together fulfilled.

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$$0 \leq \frac{1}{r}, \frac{1}{\tilde{r}} \leq \frac{1}{2}, \quad 0 < \frac{1}{q} \leq \frac{1}{2}, \quad 0 < \frac{1}{\tilde{q}} \leq 1, \qquad \text{(a, b, c, d)}$$

$$\frac{1}{q} + \frac{1}{\tilde{q}} < 1, \quad \frac{d-2}{d} \leq \frac{r}{\tilde{r}} \leq \frac{d}{d-2}, \qquad \text{(e, f)}$$

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} - s^*, \quad \frac{1}{\tilde{q}} + \frac{d}{\tilde{r}} < \frac{d}{2}, \qquad \text{(g, h)}$$

$$\frac{1}{q} + \frac{d}{r} < \frac{d}{2}, \quad \frac{2}{q} + \frac{d}{r} + \frac{2}{\tilde{q}} + \frac{d}{\tilde{r}} = d. \qquad \text{(i, j)}$$

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Case $d \geq 3$: Conditions (e), (i) and (j) follow from (g), (17) and the choice of $s^* = \frac{\alpha d-(4-2b)}{2\alpha}$. Rewriting all others in terms of $\frac{1}{r}$ leads to:

$$0 \leq \frac{1}{r} \leq \frac{1}{2}, \quad \frac{d-2b}{2d(\alpha+1)} < \frac{1}{r} < \frac{d-b}{d(\alpha+1)}, \qquad \text{(a, b)}$$

$$\frac{2-\alpha-b}{d\alpha} \leq \frac{1}{r} < \frac{2-b}{d\alpha}, \quad \frac{2-\alpha b-b}{d\alpha(\alpha+1)} < \frac{1}{r} \leq \frac{2-b}{d\alpha}, \qquad \text{(c, d)}$$

$$\frac{d-2}{\alpha(d-2)+2(d-1)}\left(\frac{d-b}{d}\right) < \frac{1}{r} < \frac{d-b}{2(d-1)+d\alpha}, \qquad \text{(f)}$$

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We can rearrange and narrow the previous set of inequalities as:

$$\max\left\{\frac{2-\alpha-b}{d\alpha}, \frac{(d-2)(d-b)}{\alpha d(d-2)+2d(d-1)}, \frac{\alpha(d-b)-(2-b)}{d\alpha(\alpha+1)}\right\} < \frac{1}{r} < \min\left\{\frac{2-b}{d\alpha}, \frac{d-b}{2(d-1)+d\alpha}\right\}. \quad (18)$$

Under the assumptions on α in (2) and $0 < b < \min\{2, d\}$, one can check that the first term in the above chain of inequalities is always not greater than the last one. This completes the proof of the theorem in the higher dimensions setting.

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Case $d = 1, 2$: We drop the inequalities (16) and notice that as a consequence of the Strichartz estimate one earns

$$0 < \frac{1}{q} \leq \frac{1}{4}, \quad \text{for} \quad d = 1, \quad 0 < \frac{1}{q} < \frac{1}{2}, \quad \text{for} \quad d = 2.$$

This implies that the interval (18) reduces, for $d = 1$, to

$$\max\left\{\frac{4-\alpha-2b}{2\alpha}, \frac{d\alpha-(2-b)-\alpha b}{\alpha(\alpha+1)}\right\} < \frac{d}{r} < \min\left\{\frac{2-b}{\alpha}, \frac{d-b}{(\alpha+1)}\right\}$$

and, for $d = 2$, to

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Proceeding as above, we see that the above set is non-empty for any $\alpha > 0$. The proof of the lemma is now completed.

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Proceeding as above, we see that the above set is non-empty for any $\alpha > 0$. The proof of the lemma is now completed.

Remark 2

The previous theorem remains valid also if we select $\frac{4-2b}{d} < \alpha < \alpha^*_\sigma(d)$*, with*

$$\alpha^*_\sigma(d) = \frac{4-2b}{d-2\sigma} \tag{19}$$

for $s^* < \sigma < 1$. *In fact we used the upper bound* $\alpha^*_1(d)$*, defined as in* (2)*, however the simple inequality* $\alpha^*_\sigma(d) < \alpha^*_1(d)$ *guarantees that the above proof can be exactly repeated with minor modifications of the involved inequalities and intervals.*

Finally, we need the following proposition, that is a direct outcome the previous steps. We have:

Proposition 3

For any $d \geq 1$, $0 < b < \min\{2, d\}$ and any α satisfying the assumption (2) there exist, for $i = 1, 2$, real numbers $s^* < \sigma_i \leq 1$, at least a pair $(q_i, r_i) \in \mathbb{R}^2_+$ and a corresponding acceptable pair $(\tilde{q}_i, \tilde{r}_i)$ such that

$$\frac{1}{\tilde{r}_i'} = \frac{\alpha+1}{r_i} + \frac{1}{\gamma_i}, \qquad \frac{1}{\tilde{q}_i'} > \frac{\alpha+1}{q_i}, \tag{20}$$

for some $\gamma_1, \gamma_2 > 0$ such that $\gamma_1 b < d$ and $\gamma_2 b > d$. In addition the solution to (1) fulfils both the homogeneous (12) and the extended inhomogeneous (14) Strichartz estimates.

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Sketch of the proof. We look at the case $d \geq 3$ (the cases $d = 1, 2$ can be handled in a similar manner). We argue by a continuity argument as follows. Fix $(\frac{1}{q}, \frac{1}{r}, \frac{1}{\tilde{q}}, \frac{1}{\tilde{r}}, s^*)$ as in the main Theorem and we look for $(\frac{1}{q+\epsilon_1}, \frac{1}{r\pm\epsilon_2}, \frac{1}{\tilde{q}_{\epsilon_1,\epsilon_2}}, \frac{1}{\tilde{r}}, \frac{b}{d} \pm \epsilon_3, \sigma^{\pm}_{\epsilon_1,\epsilon_3})$ that satisfy the conditions of the corollary, for some $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ small enough and $\tilde{q}_{\epsilon_1,\epsilon_2}, \sigma^{\pm}_{\epsilon_1,\epsilon_3}$ properly chosen. By our choice it is clear that $\lim_{\epsilon_1,\epsilon_3\to 0} \sigma^{\pm}_{\epsilon_1,\epsilon_3} = s^*$ (with $\sigma^+ > \sigma^-_{\epsilon_1,\epsilon_3} > s^*$) and $\lim_{\epsilon_1,\epsilon_2\to 0} \tilde{q}_{\epsilon_1,\epsilon_2} = \tilde{q}$. We have also

$$0 \leq \frac{1}{r\pm\epsilon_2}, \frac{1}{\tilde{r}} \leq \frac{1}{2}, \quad 0 < \frac{1}{q+\epsilon_1} \leq \frac{1}{2}, \quad 0 < \frac{1}{\tilde{q}_\epsilon} \leq 1,$$

$$\frac{1}{q+\epsilon_1} + \frac{1}{\tilde{q}_{\epsilon_1,\epsilon_2}} \mp \frac{\epsilon_2}{r^2 \pm r\epsilon_2} = \frac{d}{2}\left(\frac{1}{\tilde{r}'} - \frac{1}{r}\right) = \beta, \quad \beta > 0,$$

$$\frac{1}{\tilde{r}'} = \frac{\alpha+1}{r\pm\epsilon_2} + \frac{b}{d} \pm \epsilon_3, \quad \frac{1}{q+\epsilon_1} < \frac{1}{\tilde{q}'_{\epsilon_1,\epsilon_2}}, \quad \frac{d-2}{d} \leq \frac{r}{\tilde{r}} \leq \frac{d}{d-2}.$$

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Let us focus on the fifth identity given above. We have

$$
\begin{aligned}
\frac{1}{\tilde{r}'} = \frac{\alpha+1}{r \pm \epsilon_2} + \frac{b}{d} \pm \epsilon_3 &= (\alpha+1)\left(\frac{1}{r} \mp \frac{\epsilon_2}{r^2 \pm r\epsilon_2}\right) + \frac{b}{d} \pm \epsilon_3 \\
&= \left(\frac{\alpha+1}{r} + \frac{b}{d}\right) \mp \frac{(\alpha+1)\epsilon_2}{r^2 \pm r\epsilon_2} \pm \epsilon_3 .
\end{aligned}
$$

Moreover, by the forth identity insetead we achieve

$$
\begin{aligned}
&\frac{\alpha+1}{q+\epsilon_1} + \frac{1}{\tilde{q}_{\epsilon_1,\epsilon_2}} \\
&= (\alpha+1)\left(\frac{1}{q} - \frac{\epsilon_1}{q^2+q\epsilon_1}\right) + \left(\beta - \frac{1}{q} + \frac{\epsilon_1}{q^2+\epsilon_1} \pm \frac{\epsilon_2}{r^2 \pm r\epsilon_2}\right) \\
&= \frac{\alpha+1}{q} - \frac{\epsilon_1(\alpha+1)}{q^2+q\epsilon_1} + \frac{1}{\tilde{q}} + \frac{\epsilon_1}{q^2+q\epsilon_1} \pm \frac{\epsilon_2}{r^2 \pm r\epsilon_2} \\
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Let us focus on the fifth identity given above. We have

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$$= 1 - \frac{\epsilon_1\alpha}{q^2+q\epsilon_1} \pm \frac{\epsilon_2}{r^2 \pm r\epsilon_2} < 1,$$

is satisfied by a continuity argument provided that $\epsilon_1 > 0$ and $\epsilon_2 > 0$ is small enough. It conclusion, it remains to pick up now

$$\left(\frac{1}{q_1}, \frac{1}{r_1}, \frac{1}{\tilde{q}_1}, \frac{1}{\tilde{r}_1}, \frac{1}{\gamma_1}, \sigma_1\right) = \left(\frac{1}{q+\epsilon_1}, \frac{1}{r+\epsilon_2}, \frac{1}{\tilde{q}_{\epsilon_1,\epsilon_2}}, \frac{1}{\tilde{r}}, \frac{b}{d} + \epsilon_3, \sigma^+_{\epsilon_1,\epsilon_3}\right)$$

and

$$\left(\frac{1}{q_2}, \frac{1}{r_2}, \frac{1}{\tilde{q}_2}, \frac{1}{\tilde{r}_2}, \frac{1}{\gamma_2}, \sigma_2\right) = \left(\frac{1}{q+\epsilon_1}, \frac{1}{r-\epsilon_2}, \frac{1}{\tilde{q}_{\epsilon_1,\epsilon_2}}, \frac{1}{\tilde{r}}, \frac{b}{d} - \epsilon_3, \sigma^-_{\epsilon_1,\epsilon_3}\right).$$

Then proof of the proposition is completed.

In a similar way we can prove the following

Corollary 1

Let $d \geq 3$, $\frac{4-2b}{d} \leq \alpha < \alpha^*_\sigma(d)$, $0 < b < \min\{2, d\}$ *be fixed with* $s^* = \frac{\alpha d-(4-2b)}{2\alpha}$, $s^* \leq \sigma \leq 1$ *and* $k > 0$ *or* $|k| << 1$. *Then there exists* $\theta \in (0,1)$ *and* $(q_{\theta,i}, r_{\theta,i}\tilde{q}_{\theta,i}, \tilde{r}_{\theta,i})$, *for* $i = 1, 2$, *such that:*

$$\frac{1}{\tilde{q}'_\theta} = \frac{(\alpha+1)(1-\theta)}{q_{\theta,i}}, \tag{21}$$

$$\frac{1}{\tilde{r}'_{\theta,i}} = (\alpha+1)\left[\frac{1-\theta}{r_{\theta,i}} + \frac{\theta}{p_i}\right] + \frac{1}{\gamma_i}, \tag{22}$$

with $\gamma_1, \gamma_2 > 0$ *such that* $\gamma_1 b < d$, $\gamma_2 b > d$ *and* $p_1, p_2 \in (2, 2d/(d-2\sigma))$ *(* $p_1, p_2 \in (2, \infty)$ *for* $d = 1, 2$*).*

We can summarize, then, the results achieved till now as follows.

Proposition 4

Let $d \geq 1$, $\frac{4-2b}{d} < \alpha < \alpha^*_\sigma(d)$ be fixed. Then there exist $(q, r, \tilde{q}, \tilde{r}) \in \mathbb{R}^4_+$ and $\frac{\alpha d-(4-2b)}{2\alpha} \leq \sigma < 1$ such that one has the following estimate

$$\|e^{t\Delta_x} f\|_{L^q_t L^r_x} + \left\| \int_0^t e^{(t-\tau)\Delta_{x,y}} F(\tau) d\tau \right\|_{L^q_t L^r_x} \leq C \left(\|f\|_{H^\sigma_x} + \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} \right) \tag{23}$$

assuming that $(q, r, \tilde{q}, \tilde{r})$ are given as in Proposition 3 or in Corollary 1.

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Outline

Sketch of the proof: local well-posedness. We set $I = (0, T)$, use the notation $L^q_T(X) = L^q((0,T);X)$ and introduce the space

$$\|w\|_{X_I} = \sup_{(q,r)\in\mathcal{A}^\sigma} \left\{ \|w\|_{L^q_I L^r_x} \right\}. \tag{24}$$

clearly

$$\|w\|_{X_I} \leq \|w\|_{X_I(\{|x|\leq 1\})} + \|w\|_{X_I(\{|x|>1\})}, \tag{25}$$

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We will perform now a contraction argument. Namely, let be defined the integral operator associated to the Cauchy problem (1),

$$\mathcal{T}_f(u) = e^{t\Delta_x} f + k \int_0^t e^{(t-\tau)\Delta_x}(|x|^{-b}|u|^\alpha u)(\tau, x)\, d\tau. \qquad (26)$$

One needs to show that for any (q, r) as in Proposition 4, if $f \in H_x^\sigma$, then there exist a $T = T(\|f\|_{H_x^\sigma}) > 0$ and a (unique)

$$u(t, x) \in X_I,$$

satisfying the property

$$\mathcal{T}_f(u(t)) = u(t), \qquad (27)$$

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For simplicity, we split the proof in three different steps.
Step One: for any $f \in H^\sigma_x$, *there exist* $T = T\left(\|f\|_{H^\sigma_x}\right) > 0$ *and* $R = R\left(\|f\|_{H^\sigma_x}\right) > 0$ *such that* $\mathcal{T}_f(B_{X_I}(0,R)) \subset B_{X_I}(0,R)$, *for any* $T' < T$.

By (23) in Proposition 4 combined with (25), we have

$$\|\mathcal{T}_f u\|_{X_I} \lesssim \|f\|_{H^{\sigma_1}_x} + \|f\|_{H^{\sigma_2}_x} \quad (28)$$
$$+\||x|^{-b}|u|^\alpha u\|_{L_I^{\tilde{q}_1'} L^{\tilde{r}_1'}(\{|x|\le 1\})} + \||x|^{-b}|u|^\alpha u\|_{L_I^{\tilde{q}_2'} L^{\tilde{r}_2'}(\{|x|>1\})},$$

with $(\tilde{q}_1, \tilde{r}_1)$ and $(\tilde{q}_2, \tilde{r}_2)$ as in Proposition 3. We notice that

$$|x|^{-b} \in L^{\gamma_1}(\{|x| \le 1\}), \quad \text{if} \quad \frac{1}{\gamma_1} > \frac{b}{d}$$

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Thus, by Hölder inequality and (20), the nonlinear terms on the r.h.s. of (28) can be bounded as

$$\begin{aligned} & \||x|^{-b}|u|^{\alpha}u\|_{L_I^{\tilde{q}_1'}L^{\tilde{r}_1'}(\{|x|\le 1\})} + \||x|^{-b}|u|^{\alpha}u\|_{L_I^{\tilde{q}_2'}L^{\tilde{r}_2'}(\{|x|>1\})} \\ & \qquad \lesssim \left\| \||x|^{-b}\|_{L^{\gamma_1}(\{|x|\le 1\})} \|u\|^{\alpha+1}_{L^{r_1}(\{|x|\le 1\})} \right\|_{L_I^{\tilde{q}_1'}} \\ & \qquad + \left\| \||x|^{-b}\|_{L^{\gamma_2}(\{|x|>1\})} \|u\|^{\alpha+1}_{L^{r_2}(\{|x|>1\})} \right\|_{L_I^{\tilde{q}_2'}} \\ & \lesssim T^{\kappa_1(\alpha)}\|u\|^{\alpha+1}_{L_I^{q_1}L^{r_1}(\{|x|\le 1\})} + T^{\kappa_2(\alpha)}\|u\|^{\alpha+1}_{L_I^{q_2}L^{r_2}(\{|x|>1\})} \\ & \qquad \lesssim T^{\kappa(\alpha)}\|u\|^{\alpha+1}_{X_I}, \end{aligned} \tag{29}$$

with $T^{\kappa(\alpha)} = \max\{T^{\kappa_1(\alpha)}, T^{\kappa_2(\alpha)}\}$. By selecting $\sigma = \sigma_1$ and using the natural embedding $H_x^{\sigma} \subseteq H_x^{\sigma_2}$ we achieve then

$$\|\mathcal{T}_f u\|_{X_I} \lesssim \|f\|_{H_x^{\sigma}} + T^{\kappa(\alpha)}\|u\|^{\alpha+1}_{X_I} \tag{30}$$

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\lesssim T^{\kappa_1(\alpha)}\|u\|^{\alpha+1}_{L_I^{q_1}L^{r_1}(\{|x|\le 1\})} + T^{\kappa_2(\alpha)}\|u\|^{\alpha+1}_{L_I^{q_2}L^{r_2}(\{|x|>1\})} \\
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Step Two: let $T, R > 0$ as in the previous step then there exists $\overline{T} = \overline{T}(\|f\|_{H_x^\sigma}) < T$ such that $\mathcal{T}_f$ is a contraction on $B_{X_{\bar{I}}}(0, R)$, equipped with the norm $\|\cdot\|_{X_{\bar{I}}}$.

Given any $v_1, v_2 \in B_{X_I}(0, R)$ we attain, by an use of (23) and (25), the following:

$$\|\mathcal{T}_f v_1 - \mathcal{T}_f v_2\|_{X_I} \lesssim \||x|^{-b}(v_1|v_1|^\alpha - v_2|v_2|^\alpha)\|_{L_I^{\tilde{q}_1'} L^{\tilde{r}_1'}(\{|x|\leq 1\})} + \||x|^{-b}(v_1|v_1|^\alpha - v_2|v_2|^\alpha)\|_{L_I^{\tilde{q}_2'} L^{\tilde{r}_2'}(\{|x|>1\})}.$$

Then arguing as above one get finally

$$\|\mathcal{T}_f v_1 - \mathcal{T}_f v_2\|_{X_I} \lesssim T^{\kappa(\alpha)}\left(\|v_1\|_{X_I}^\alpha + \|v_2\|_{X_I}^\alpha\right)\|v_1 - v_2\|_{X_I},$$

and this completes the proof of this step.

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Step Three: the solution exists and is unique in $X_{\bar{T}}$, where $\bar{T}$ is as in the above step.

We are in position to show existence and uniqueness of the solution by applying the contraction principle to the map $\mathcal{T}_f$ defined on the complete metric space $B_{X_{\bar{T}}}(0, R)$, equipped with the topology induced by $\| \cdot \|_{X_{\bar{T}}}$.

Global well-posedness: the proof can be inferred similary to the one we performed above just with small modifications and by using now the Corollary 1. This result allows to extend globally the solution if $k > 0$ or $k < 0$ and so that $k << 1$ by means of the conservation law (3). So we skip.

Step Three: the solution exists and is unique in $X_{\bar{l}}$, where $\bar{l}$ is as in the above step.

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Open problems:

1. A paper by R. J. Taggart (2010) suggests that, in the case of unitary operators associated to the solution of an evolution equation (Schrödinger equation, for instance), the extended inhomogeneous Strichartz estimates can be obtained in the Lorentz spaces setting. We would consider again, if it is possible, to have an analogous for the self-adjoint operator $e^{t\Delta_x}$.
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Thank You for your attention.